

Proportional justified representation

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ABSTRACT

The goal of multi-winner elections is to choose a fixed-size committee based on voters' preferences. An important concern in this setting is *representation*: large groups of voters with cohesive preferences should be adequately represented by the election winners. In an influential paper, Aziz et al. [1] proposed two axioms that aim to capture this idea: *justified representation (JR)* and its strengthening *extended justified representation (EJR)*. We observe that EJR is incompatible with the highly desirable *Perfect Representation (PR)* criterion, and propose a relaxation of EJR, which we call *Proportional Justified Representation (PJR)*. PJR is more demanding than JR, but, unlike EJR, it is compatible with PR, as well as with a stronger variant of this axiom, which we term *Fractional Perfect Representation (FPR)*. Moreover, just like EJR, PJR can be used to characterise the classic Proportional Approval Voting (PAV) rule in the class of weighted PAV rules. On the other hand, we show that EJR provides stronger guarantees with respect to average voter satisfaction than PJR does.

1. Introduction

Decision-making based on the aggregation of possibly conflicting preferences is a central problem in the field of social choice, and as such it has received a considerable amount of attention from artificial intelligence researchers [2]. The most common preference aggregation scenario is the one where a single candidate has to be selected. However, there are also many applications where the goal is to select a fixed-size set of alternatives: examples range from choosing a parliament or a committee [3,4] to identifying a set of plans [5], allocating resources [5,6], shortlisting candidates for a job or an award [4,7], picking movies to be shown on a plane [4], selecting validators in a blockchain consensus protocol [8,9], putting together a slate of opinions on a controversial issue [10], or deciding on trees to be removed from a forest [11]. Multi-winner elections can also be used in a range of machine learning tasks,

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¹ This work was developed while Edith Elkind, Martin Lackner, and Piotr Skowron were at the University of Oxford, United Kingdom.

² Retired

such as recommender systems, dataset construction, or core-set selection (also referred to as instance selection); we discuss several such applications in more detail towards the end of this section.

Recently, the computational complexity of multi-winner voting rules and their normative properties have been actively explored by the artificial intelligence research community. Faliszewski et al. [12] give an overview of algorithmic and axiomatic challenges posed by multi-winner elections; the survey of Lackner and Skowron [13] focuses specifically on approval-based multi-winner voting.

Multi-winner voting rules are often applied in scenarios in which the set of winners needs to represent the different opinions or preferences of the agents participating in the election. Thus, it is important to formulate axioms that capture our intuition about what it means for a set of winners to provide a faithful representation of voters' preferences [3,14,15].

We build on the influential work of Aziz et al. [1], who have proposed two representation axioms for approval-based multi-winner voting, namely, *justified representation* (JR) and *extended justified representation* (EJR). Intuitively, JR requires that a large enough group of voters with similar preferences is allocated at least one representative; EJR says that if this group is large enough and cohesive enough, it deserves not just one, but several representatives (see Section 2 for formal definitions). Similar axioms have been proposed for multi-winner voting rules with ranked ballots [4,14]. Aziz et al. [1] show that for every collection of ballots there is a winning set that provides EJR; they then explore a number of popular multi-winner voting rules and show that several of these rules satisfy JR, but only one of them (namely, Proportional Approval Voting) satisfies EJR.

Our Contribution

We approach the issue of proportional representation in approval-based multi-winner voting from a different perspective. We start by formulating an axiom that we call *Perfect Representation* (PR) (Section 3), which says that if a given instance admits a 'perfect solution' (all voters are represented, and each winner represents the same number of voters), then we expect a voting rule to output such a solution. This axiom is very appealing in parliamentary elections and similar applications of multi-winner voting. However, it turns out to be incompatible with EJR: there is an election where these two axioms correspond to disjoint sets of winning committees.

Motivated by this result, we propose a relaxation of EJR, which we call *Proportional Justified Representation* (PJR) (Section 4). PJR is more demanding than JR, but, unlike EJR, it is compatible with perfect representation. Further, we show that, under a mild assumption, a committee that provides PJR can be computed in polynomial time; specifically, we demonstrate that a well-studied, efficiently-computable voting rule (namely, the Greedy Monroe rule) satisfies PJR if the committee size k divides the number of voters n . Moreover, just like EJR, PJR can be used to characterise the classic Proportional Approval Voting (PAV) within the class of weighted PAV rules [16].

However, we then show that the additional flexibility supplied by PJR comes at a cost (Section 5): we define a measure of average voter satisfaction and show that EJR provides much stronger guarantees with respect to this measure than PJR does.

Also, we formulate a stronger version of the PR axiom (Section 6): while the original axiom only applies when the target committee size k divides the number of voters n , the new axiom, which we call *Fractional Perfect Representation* (FPR), allows for non-integer vote allocation and applies to all values of k and n . Interestingly, FPR is compatible with PJR; indeed, we identify a voting rule that satisfies both axioms.

We conclude the paper by discussing the implications of our results and indicating directions for future work.¹

Fig. 1 shows the relations between the different axioms considered in this paper, including both those proposed here and those proposed by other authors. Relations are established at the level of committees: whenever there is a link between two axioms, this means that if a committee satisfies the upper axiom, then it also satisfies the lower axiom.

Related and Subsequent Work

In the following, we give an overview of the research inspired by the conference version of our paper and related lines of research; for details, the reader is encouraged to consult the survey by Lackner and Skowron [13].

It is interesting to note that verifying whether a given committee satisfies PJR is computationally hard; Aziz et al. [18] show that this problem is coNP-complete. Further, Bredereck et al. [19] prove that counting the number of committees satisfying PJR is #P-hard.

After our conference paper, several axioms that strengthen PJR have been proposed in the literature. Peters and Skowron [20] introduced another property of voting rules, which they term *priceability*. This property is a strengthening of PJR that does not rely on cohesive groups (cf. Definition 1). Similarly, Aziz and Lee [21] propose a strengthening of PJR, called *Inclusion Proportionality for Solid Coalitions* (IPSC). In doing so, they build on their earlier work [22], which puts forward an axiom for weak preferences, called *Generalized PSC*. The generalized PSC axiom generalizes both the classical *Proportionality for Solid Coalitions* (PSC) due to Dummett [14] for strict preferences and PJR for approval voting. We discuss these axioms in detail in Section 6. Peters et al. [23] proposed a strengthening of EJR by relaxing the notion of a cohesive group (see Section 2 for formal definitions). They call this strengthening *fully justified representation* (FJR). Very recently, Kalayci et al. [24] applied the ideas introduced by Peters et al. [23] to define an intermediate axiom between PJR and FJR (but incomparable to EJR), which they called *Full Proportional Justified Representation*.

¹ The preliminary version of this paper appeared in the Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI) in 2017 [17]. The main additions compared to the conference version are the introduction of the FPR axiom and the inclusion of additional examples, figures, and discussion. Other additions include complete proofs of Theorems 1 and 4, and Propositions 4 and 5. We also show that from the point of view of the maximum number of unrepresented voters in an ℓ -cohesive group of voters (see Section 2 for the definition of ℓ -cohesive group of voters), there is no difference between JR, PJR, and EJR.

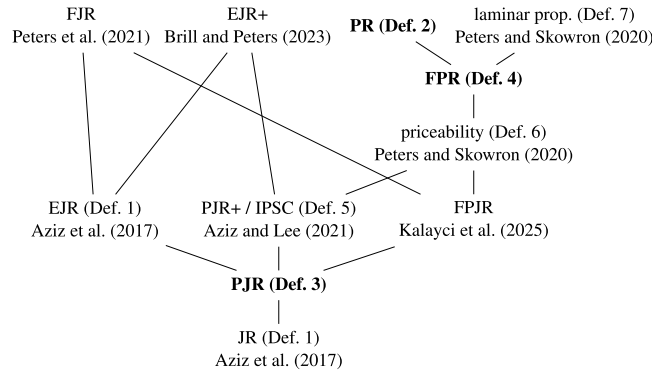


Fig. 1. Hasse diagram for the representation axioms considered in this paper and other state-of-the-art papers. The lines indicate that if a committee satisfies the upper axiom, it also satisfies the lower axiom. Axioms in bold are contributions of this paper.

(FPJR). Finally, Brill and Peters[25] have recently proposed strengthenings of PJR and EJR, which they have called, respectively, PJR+ and EJR+. A salient property of PJR+ and EJR+ is that, in contrast to PJR and EJR, it is possible to verify whether a committee satisfies PJR+ or EJR+ in polynomial time.

Peters [26] showed that even very weak notions of proportionality (and, in particular, PJR) are incompatible with a natural notion of strategyproofness.²

Approval-based multi-winner voting is closely related to the *apportionment* problem [27,28], which arises, e.g., in parliamentary elections when distributing seats to parties. This relation has been explored in depth by Brill et al. [29], including the impact of PJR in this setting. An important result proved in this paper is that any multi-winner voting rule that satisfies PJR induces an apportionment method that satisfies lower quota. Moreover, Brill et al. [30] consider PJR in a setting that generalizes apportionment but is still somewhat less expressive than the standard approval-based multi-winner setting. We formally define apportionment problems and discuss their relationship with FPR in Section 6.

The concept of PJR has also been used in the context of *participatory budgeting*. Aziz et al. [31] extend PJR to this setting and propose several related axioms. Furthermore, PJR has also been adapted to approval-based elections with a variable number of winners [32] (this model differs from the standard model in that the size of the committee is not fixed), and to sub-committee voting [33].³

Another important contribution of this paper is the concept of *average satisfaction* (Section 5). This concept, together with the related concept of *proportionality degree*, has been refined and used in several works to evaluate the performance of various multi-winner voting rules [18,34,35].

An experimental evaluation of PJR and related notions of proportionality has been performed by Bredereck et al. [19]. Their numerical simulations show that in a typical election many committees satisfy even EJR (and thus PJR), and therefore these proportionality requirements are not by themselves sufficient to select committees.

We defined PJR in terms of the *Hare quota* $\frac{n}{k}$. Recently, Casey and Elkind [36] considered a strengthening of PJR (and other justified representation axioms) obtained by replacing the Hare quota with the *Droop quota* $\frac{n}{k+1}$, and observed that many voting rules that satisfy PJR also satisfy (or can be modified to satisfy) the Droop version of this axiom.

1.1. Fair representation and AI

Before presenting our formal model, we would like to offer a few examples of AI applications that make use of fair representation ideas.

The first application we would like to discuss is that of recommender systems, which are “one of the most popular applications of artificial intelligence” [37]. A typical recommender system collects data about the preferences of the users, such as ratings or clicks on links, and uses this data to make recommendations to other users. Some state-of-the-art recommender systems actively use social choice techniques [38]. In particular, some applications aim to produce recommendations for a group of users, first coming up with suggestions for each member of the group and then aggregating them into a joint recommendation. In this scenario, it may be desirable for the recommendation system to produce a set of recommendations that represent various groups of users, i.e., take into account the interests of minorities rather than focus exclusively on the interests of the majority within the group. These types of recommender

² Specifically, the notion of strategyproofness used by Peters [26] is that a rule is strategyproof if no voter with ballot A can change the outcome of the election by submitting a ballot $A' \subsetneq A$ to obtain more representatives, i.e., so that for the original committee W and the new committee W' it holds that $W \cap A \subsetneq W' \cap A$.

³ Sub-committee voting is a generalization of several preference aggregation scenarios, including single-winner voting, multi-winner voting, and multiple referenda. Briefly, the goal of a sub-committee election is to select several pairwise disjoint sub-committees, so that all voters participate in the election of all the sub-committees. Each sub-committee has a fixed size, but the sizes of different sub-committees may differ.

systems are relevant for tourism, as discussed by Streviniotis and Chalkiadakis [39], or for the generation of trending topics in a social network [40], and a number of authors have explored using the toolbox of multi-winner voting in this context [39,41,42].

The concepts of fairness and proportional representation are also receiving growing attention in the context of machine learning and its applications, where an important desideratum is to avoid minorities being misrepresented or harmed by decisions of machine learning algorithms [43,44]. Fair representation concerns also play a key role in the construction of models and datasets, where it is crucial to incorporate various data sources in a proportional way; see, for instance, the work on generative modeling [45] or core-set selection (further discussed below) [46]. A number of works translate fairness notions from the computational social choice literature to the domain of clustering [47–52]. Examples of applications in which fairness concerns are relevant in clustering range from achieving precise personalised recommendations [47] to selecting a subset of the instances in a dataset to label so as to obtain a faithful representation of the original dataset in preparation for a machine learning task [53].

Another machine learning topic where the use of representation concepts from computational social choice has been explored is core-set selection, also known as instance selection. Core-set selection [46] is a preprocessing task in machine learning (or data mining) that aims at selecting a subset of the data instances to form the training set, to be used by a machine learning algorithm. There are two main reasons for performing this task: efficiency and cleaning. Storing, preprocessing, and training models on large datasets requires costly computing resources. Thus, it is desirable to identify a subset of the original dataset that allows the machine learning task to be performed with little or no performance loss. A scenario in which this problem is particularly important is that of *class incremental learning* or *continual learning* [54]. In class incremental learning, we have a stream that, from time to time, receives a new class along with a set of instances belonging to that class. At specific time points, we need to build a classifier that can recognise all the classes seen so far. Storing all received instances would be too costly; therefore, for each class observed so far, it is necessary to select a subset of the instances of that class that will be stored. Another purpose of core-set selection is cleaning: often, the datasets used in machine learning contain noisy instances, which can lead to errors in the classifier. Removing these noisy instances can improve the performance of a machine learning algorithm.

Sánchez-Fernández et al. [55] conjecture that using representative approval-based multi-winner voting rules would allow for obtaining a representative subset of the original dataset and thereby reduce the size of the dataset with little performance cost. They establish a formal guarantee that, under certain conditions, a K Nearest Neighbours (KNN) classifier trained on a subset of the original dataset obtained by applying a multi-winner voting rule that satisfies the proportional justified representation⁴ axiom, introduced in this paper, will correctly classify the instances from the original dataset.

2. Preliminaries

Given a positive integer s , we denote the set $\{1, \dots, s\}$ by $[s]$. We consider elections with a set of voters $N = \{1, \dots, n\}$ and a set of candidates $C = \{c_1, \dots, c_m\}$. Each voter $i \in N$ submits an *approval ballot* $A_i \subseteq C$, which represents the subset of candidates that she approves of. We refer to the list $\mathcal{A} = (A_1, \dots, A_n)$ as the *ballot profile*. An *approval-based multi-winner voting rule* takes as input a tuple $(N, C, \mathcal{A}, k)$, where k is a positive integer that satisfies $k \leq |C|$ (we will refer to such tuples as *elections*), and returns a subset $W \subseteq C$ of size k , which we call the *winning set*, or *committee*. We omit N and C from the notation when they are clear from the context.

The following voting rules have received a considerable amount of attention in the literature on proportional representation [1,4,56]:

Thiele rules and Proportional Approval Voting (PAV) Every vector $\mathbf{w} = (w_1, w_2, \dots)$, where $w_1, w_2, \dots$ are non-negative reals, $w_1 = 1$ and $w_1 \geq w_2 \geq \dots$, defines a voting rule, which we will call the *w-Thiele rule*. Under this rule, if a voter approves p candidates in a committee W then she assigns a score of $w_1 + \dots + w_p$ to W ; the scores are then summed over all voters. Formally, given a ballot profile $(A_1, \dots, A_n)$ and a target number of winners k , this rule computes the *w-Thiele score* of a committee W of size k as

$$\sum_{i \in N} \sum_{j=1}^{|W \cap A_i|} w_j,$$

and returns a committee with the highest score.

A particularly important and well-studied rule in this family is Proportional Approval Voting (PAV), which uses the harmonic weight vector $(1, \frac{1}{2}, \frac{1}{3}, \dots)$.

Sequential Thiele rules and Seq-PAV For each w-Thiele rule, we define its sequential analogue *Seq-w-Thiele*. This rule proceeds in k rounds, adding one candidate to the committee in each round. Specifically, in round t it selects a candidate that provides the maximum improvement to the total w-Thiele score of the committee selected in the first $t - 1$ rounds. Formally, the Seq-w-Thiele rule starts by setting $W = \emptyset$. Then in round $t, t \in [k]$, it computes the *approval weight* of each candidate $c \in C \setminus W$ as

$$\sum_{i: c \in A_i} w_{|W \cap A_i|+1},$$

where W is the winning set after the first $t - 1$ rounds, selects a candidate in $C \setminus W$ with the highest approval weight, and adds it to W . Seq-PAV is the Seq-w-Thiele rule for the weight vector $\mathbf{w} = (1, \frac{1}{2}, \frac{1}{3}, \dots)$.

The Monroe rule For each voter $i \in N$ and each candidate $c \in C$ we write $u_i(c) = 1$ if $c \in A_i$ and $u_i(c) = 0$ if $c \notin A_i$. Given a committee $W \subseteq C$ of size k , we say that a mapping $\pi : N \rightarrow W$ is *valid* if it satisfies $|\pi^{-1}(c)| \in \left\{ \lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil \right\}$ for each $c \in W$. The *Monroe score* of

⁴ More precisely, it suffices for the voting rule to satisfy ℓ -PJR with $\ell = \frac{k+1}{2}$ for a KNN classifier and K odd.

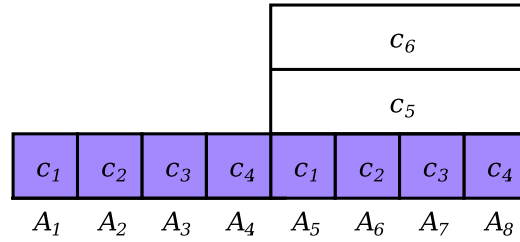


Fig. 2. Ballot profile in the proof of Theorem 2.

a valid mapping π is given by $\sum_{i \in N} u_i(\pi(i))$, and the *Monroe score* of W is the maximum score of a valid mapping from N to W . The *Monroe rule* returns a committee of size k with the maximum Monroe score.

The Greedy Monroe rule Given a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C and a target committee size k , the Greedy Monroe rule proceeds in k rounds. It maintains the set of available candidates C' and the set of unsatisfied voters N' ; initially $C' = C$ and $N' = N$. It starts by setting $W = \emptyset$. In round t , $t = 1, \dots, k$, it selects a candidate c_t from C' and a group of voters N_t from N' of size approximately $\frac{n}{k}$ (specifically, $\lceil \frac{n}{k} \rceil$ if $t \leq n - k \lfloor \frac{n}{k} \rfloor$, and $\lfloor \frac{n}{k} \rfloor$ if $t > n - k \lfloor \frac{n}{k} \rfloor$) so as to maximize the quantity $|\{i \in N_t : c_t \in A_i\}|$ over all possible choices of (N_t, c_t) . The candidate c_t is then added to W , and we set $C' = C' \setminus \{c_t\}$, $N' = N' \setminus N_t$. We say that the candidates in N_t are *assigned* to c_t . After k rounds, the rule outputs W .

All rules we have defined may have to break ties: e.g., there could be multiple committees with the maximum w-Thiele or Monroe score, and for sequential rules there may be multiple candidates that maximize the relevant quantity in a given round. Unless explicitly indicated otherwise, the results in our paper hold irrespective of the tie-breaking mechanism. For instance, when we prove that a rule satisfies a certain axiom, we show that this is true for all possible ways of breaking ties; conversely, when we establish that a rule violates an axiom, our counterexamples do not rely on tie-breaking.

Thiele rules and their sequential variants (and in particular PAV and Seq-PAV) were defined by Thiele [16]. The Monroe rule was proposed by Monroe [3]. Greedy Monroe is due to Skowron et al. [57] (more precisely, Skowron et al. [57] define this rule for the setting where ballots are rankings of the candidates; we adapt their definition to approval ballots). For PAV and Monroe finding a winning committee is NP-hard [58,59], whereas for Seq-PAV and Greedy Monroe winning committees can be computed in polynomial time; in fact, Seq-PAV and Greedy Monroe were originally proposed as approximation algorithms for PAV and Monroe, respectively.

We will now define the key concepts from the work of Aziz et al. [1]: *justified representation* and *extended justified representation*.

Given an election $(\mathcal{A}, k)$ with a set of voters $N = \{1, \dots, n\}$ and a positive integer $\ell \in [k]$, we say that a set of voters $N^* \subseteq N$ is ℓ -cohesive if $|N^*| \geq \ell \cdot \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq \ell$.

Definition 1. (Extended) justified representation ((E)JR) Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$. A set of candidates W is said to provide ℓ -justified representation (ℓ -JR) for $(\mathcal{A}, k)$ if there does not exist an ℓ -cohesive set of voters N^* such that $|A_i \cap W| < \ell$ for each $i \in N^*$. We say that W provides *justified representation* (JR) for $(\mathcal{A}, k)$ if it provides 1-JR for $(\mathcal{A}, k)$; it provides *extended justified representation* (EJR) for $(\mathcal{A}, k)$ if it provides ℓ -JR for $(\mathcal{A}, k)$ for all $\ell \in [k]$. An approval-based voting rule *satisfies* ℓ -JR if for every ballot profile $\mathcal{A}$ and every target committee size k , it outputs a committee that provides ℓ -JR for $(\mathcal{A}, k)$. A rule *satisfies* JR (respectively, EJR) if it satisfies ℓ -JR for $\ell = 1$ (respectively, for all $\ell \in [k]$).

By definition, EJR implies JR. Aziz et al. [1] characterise w-Thiele rules that satisfy JR, and establish that PAV is the only rule in this class that satisfies EJR. Further, they prove that the Monroe rule satisfies JR, but fails EJR, and Seq-PAV fails JR for sufficiently large values of k ; they do not consider Greedy Monroe in their work.

We use figures to represent examples of approval-based multi-winner elections, using the following convention: the columns correspond to voters, the shapes (typically, rectangles) correspond to candidates, and each voter approves all candidates whose shapes intersect her column. E.g., in Fig. 2 voter 3 approves candidate c_3 only, whereas voter 5 approves candidates c_1, c_5 , and c_6 .

3. Perfect representation

A key application of multi-winner voting is parliamentary elections, where an important goal is to select a committee that reflects as fairly as possible the different opinions or preferences that are present in a society. Fairness in this context means that each committee member should represent approximately the same number of voters, and as many voters as possible should be represented by a committee member that they approve. From this perspective, the best-case scenario is when each voter is represented by a candidate that she approves and each winning candidate represents exactly the same number of voters. Thus, we may want our voting rules to output committees with this property whenever they exist. This motivates the following definition.

Definition 2. Perfect representation (PR) Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$, such that k divides n . We say that a set of candidates W , $|W| = k$, provides *perfect representation* (PR) for $(\mathcal{A}, k)$ if it is possible to partition N into k pairwise disjoint subsets $N_1, \dots, N_k$ of size $\frac{n}{k}$ each and assign a distinct candidate from W to each of these subsets in such a way that for each $\ell \in [k]$ all voters in N_ℓ approve their assigned member of W . An approval-based voting

rule *satisfies PR* if for every profile $\mathcal{A}$ and every target committee size k , the rule outputs a committee that provides PR for $(\mathcal{A}, k)$ whenever such a committee exists.

An example of a voting rule that satisfies PR is the Monroe rule. Indeed, if k divides n , a size- k committee provides perfect representation for an n -voter ballot profile if and only if its Monroe score is n , i.e., the maximum possible score. This is not surprising because the ideas of representation captured by PR that we have discussed at the beginning of this section are similar to those expressed by Monroe [3].

We note that the PR axiom is quite demanding from a computational perspective: the problem of deciding whether there exists a committee that provides PR for a given pair $(\mathcal{A}, k)$ is NP-complete. Specifically, the hardness result is a straightforward adaptation of a proof of Procaccia et al. [59], while showing that this problem is in NP proceeds by a reduction to b-matching.

Theorem 1. *Given a ballot profile $\mathcal{A}$ and a target committee size k that divides the number of voters n , it is NP-complete to decide whether there exists a committee that provides PR for $(\mathcal{A}, k)$.*

Proof. To show that this problem is in NP, we will argue that, given a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ and a committee W of size k , where k divides n , we can reduce the problem of deciding whether W provides PR for $(\mathcal{A}, k)$ to finding a b-matching in a bipartite graph.

Recall that an instance of a b-matching problem is given by a bipartite graph G with parts L and R and edge set E , where each node $u \in L \cup R$ is associated with a capacity $\kappa(u)$, which is a non-negative integer. It is a ‘yes’-instance if there is a collection $E' \subseteq E$ of edges of G such that each node $u \in L \cup R$ is incident with exactly $\kappa(u)$ edges of E' . It is possible to check in polynomial time if a given instance of the b-matching problem is a ‘yes’-instance [60].

Now, given a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ and a committee W of size k , where k divides n , we construct a bipartite graph with parts N and W where there is an edge from $i \in N$ to $c \in W$ if and only if $c \in A_i$. We set the capacity of each node in N to 1 and the capacity of each node in W to $\frac{n}{k}$. Clearly, W provides PR if and only if our instance of b-matching is a yes-instance.

To prove NP-hardness, we modify an argument due to Procaccia et al. [59], which shows that finding a committee whose Monroe score is at least as high as a given bound is NP-hard. Specifically, Procaccia et al. [59] transform an instance of the classic NP-hard problem EXACT COVER BY 3-SETS (X3C) [61] into an approval-based multi-winner election with $n = 3k$, by mapping elements of the ground set to voters and size-3 sets to candidates, and observe that the original instance of X3C admits an exact cover if and only if the resulting election admits a committee whose Monroe score is equal to n . For such an election, the Monroe rule requires that each candidate represents exactly 3 voters, and a committee has Monroe score of n if and only if it provides PR. This proves the hardness of deciding whether there exists a committee that provides PR for a given election. For details of the reduction, we refer to the work of Procaccia et al. [59]. $\square$

Remark 1. Theorem 1 immediately implies that, unless $P = NP$, Seq-PAV and Greedy Monroe fail PR, in the sense that, for some way of breaking ties, these rules output committees that do not provide PR. It is also not hard to construct specific examples on which these rules fail PR in this sense. Both PAV and seq-PAV fail PR in an even stronger sense: for $k = 4$ the election depicted in Fig. 2 admits a unique PR committee (namely, $\{c_1, c_2, c_3, c_4\}$), but it cannot be output by PAV or seq-PAV, no matter how we break ties. In contrast, Greedy Monroe can always break ties so as to output a PR committee if it exists. Indeed, if a committee $W = \{c_1, \dots, c_k\}$ provides PR, Greedy Monroe can select c_t in step t for $t = 1, \dots, k$: this choice is consistent with the definition of the rule.

Viewed from a different perspective, PR is a rather weak axiom: it only constrains the behavior of a voting rule on inputs that admit a committee that provides PR. In particular, this axiom has no bite if k does not divide n . Also, unlike EJR, PR does not engage with the idea that a voter may benefit from being represented by more than one candidate. Thus, we may want a voting rule to satisfy both PR and another representation axiom, such as, e.g., EJR. However, this turns out to be impossible: PR and EJR are incompatible.

Theorem 2. *There exists a ballot profile $\mathcal{A}$ and a target committee size k such that the set of committees that provide PR for $(\mathcal{A}, k)$ is non-empty, but none of the committees in this set provides EJR.*

Proof. Let $C = \{c_1, \dots, c_6\}$, and consider a ballot profile $\mathcal{A} = (A_1, \dots, A_8)$ where $A_i = \{c_i\}$, $A_{i+4} = \{c_i, c_5, c_6\}$ for $i = 1, \dots, 4$ (see Fig. 2). Let $k = 4$. Observe that $W = \{c_1, c_2, c_3, c_4\}$ is the unique committee of size 4 that provides PR for $(\mathcal{A}, 4)$. However, W fails to provide EJR: $\{5, 6, 7, 8\}$ is a 2-cohesive set of voters, but each of these voters only approves one candidate in W (all committees that provide EJR for this instance contain c_5 or c_6). $\square$

This motivates the following question: can we find a weakening of the EJR axiom that still provides meaningful guarantees to large cohesive groups of voters, yet is compatible with PR? We address this question in the next section.

Remark 2. PR is also incompatible with the core [1,20], a concept inherited from cooperative game theory. A size- k committee W is in the core of an election $(\mathcal{A}, k)$ if there does not exist a set of voters $N^* \subseteq N$ and a set of candidates $D \subseteq C$ with $|D| \leq |N^*| \cdot \frac{k}{n}$ such that $|D \cap A_i| > |W \cap A_i|$ for each voter i in N^* . It is an open problem whether the core of every multiwinner election with approval ballots is non-empty, but it is known that every committee in the core provides EJR [1]. The example used in the proof of Theorem 2 also shows the incompatibility between PR and the core. The committee $W = \{c_1, c_2, c_3, c_4\}$ is not in the core because each of the voters in the subset $\{5, 6, 7, 8\}$ has two of their preferred candidates in $\{c_5, c_6\}$ and only one in W . In this election the core is not empty: for instance, $\{c_1, c_2, c_5, c_6\}$ is in the core.

c_5				
c_4				
c_3				
c_2				
c_1				
d_1	d_2	d_3	d_4	d_5
A_1	A_2	A_3	A_4	A_5

Fig. 3. Ballot profile in Example 1.

3.1. Further examples

In this section, we further discuss the significance of the PR axiom. To this end, we consider two different scenarios and explore the implications of this axiom in each case. It will be convenient to illustrate our arguments by means of the following example, which is a particular instance of a family of elections discussed by [Lackner and Skowron [13], p. 55] (we will also use this election family in Example 6).

Example 1. Let $C = \{c_1, \dots, c_5, d_1, \dots, d_5\}$, and consider a ballot profile $\mathcal{A} = (A_1, \dots, A_5)$ where $A_i = \{c_1, c_2, c_3, c_4, c_5, d_i\}$ for $i = 1, \dots, 5$ (see Fig. 3). Suppose that the target committee size is 5. Consider the following two committees: $W^c = \{c_1, c_2, c_3, c_4, c_5\}$ and $W^d = \{d_1, d_2, d_3, d_4, d_5\}$. We will refer to W^c as the ‘consensus committee’ (and to $c_1, \dots, c_5$ as the ‘consensus candidates’) and to W^d as the ‘disagreement committee’ (and to $d_1, \dots, d_5$ as the ‘disagreement candidates’). Then both W^c and W^d provide PR, but only W^c provides EJR (to see that W^d fails to provide EJR, note that the set of all voters forms a 5-cohesive group, so EJR demands that at least one voter approves 5 candidates in the selected committee).

3.1.1. Voting on issues

Consider a multiwinner election where both voters and candidates express their opinions on a number of “yes/no” issues, and the eventual decision on each issue is taken by the selected committee via a majority vote. In this case, it is desirable that, on as many issues as possible, the committee decisions follow the opinion of the majority of the voters. We note that this setting has been studied by the (computational) social choice community (see, e.g., [62–64]).

It is known that even if there are just two candidates that disagree on every issue and the goal is to select one of them (i.e., $k = 1$), each voter selects the candidate who agrees with her on a majority of issues, and the candidate selected by a majority of voters is declared the election winner, the winner may disagree with the opinion of the majority of voters on each issue; this phenomenon is known as the Ostrogorski paradox (see, e.g., [65,66]).

In what follows, we consider a multiwinner variant of this paradox that occurs when voters select candidates that match perfectly with their preferences on the issues considered, and the committee is selected according to certain notions of representation; yet, some of the decisions made by the committee fail to follow the opinion of the majority of the voters.

We start by showing that, in this scenario, there is no guarantee that the consensus committee represents the voters better than the disagreement committee. To see why, we construct an example where each voter i is closer to the opinions of candidate d_i than to those of the consensus candidates. We note that in many situations this is to be expected: e.g., the members of the disagreement committee could be the voters themselves, or, more broadly, for each voter there may be a “niche” candidate that closely matches their preferences as well as a slate of broadly popular candidates that she mostly agrees with. Accordingly, in our example the decisions of the consensus committee exhibit less agreement with the opinions of the majority of the voters compared to those of the disagreement committee, and the disagreement committee is, in fact, more representative than the consensus committee. The following example instantiates this idea.

Example 2. Consider a 5-voter election with candidate set $\{c_1, \dots, c_5, d_1, \dots, d_5\}$, where the goal is to select a 5-member committee that will decide on 15 “yes/no” issues. Both voters and candidates have their own preferences over issues. Specifically, for each $i = 1, \dots, 5$ both voter i and candidate d_i have the same preferences over issues, as indicated in Table 1. In contrast, for each $i = 1, \dots, 5$ candidate c_i prefers “yes” on all issues.

Table 1
Preferences over issues of voters and disagreement candidates in [Example 2](#).

Issue	Voter i /Candidate d_i				
	1	2	3	4	5
1	“no”	“no”	“no”	“yes”	“yes”
2	“no”	“yes”	“yes”	“no”	“no”
3	“no”	“yes”	“yes”	“yes”	“yes”
4	“yes”	“no”	“yes”	“yes”	“yes”
5	“yes”	“no”	“yes”	“yes”	“yes”
6	“yes”	“yes”	“no”	“yes”	“yes”
7	“yes”	“yes”	“no”	“yes”	“yes”
8	“yes”	“yes”	“yes”	“no”	“yes”
9	“yes”	“yes”	“yes”	“no”	“yes”
10	“yes”	“yes”	“yes”	“yes”	“no”
11	“yes”	“yes”	“yes”	“yes”	“no”
12	“yes”	“yes”	“yes”	“yes”	“yes”
13	“yes”	“yes”	“yes”	“yes”	“yes”
14	“yes”	“yes”	“yes”	“yes”	“yes”
15	“yes”	“yes”	“yes”	“yes”	“yes”

Table 2
Preferences over issues of voters in [Example 3](#).

Issue	Voters				
	{1, 2, 3}	{4, 5, 6}	{7}	{8}	{9}
1	“yes”	“no”	“yes”	“yes”	“no”
2	“yes”	“no”	“yes”	“no”	“yes”
3	“yes”	“no”	“no”	“yes”	“yes”

If each voter approves all candidates that agree with them on at least 80% of the issues, the approvals of the voters are as in [Example 1](#): each voter i approves d_i and all consensus candidates $c_1, \dots, c_5$.

Now, if the disagreement committee W^d is elected, its decisions coincide with those of the majority of the voters on each issue. However, the consensus committee W^c disagrees with the majority of the voters on issues 1 and 2. Thus, in this example, the disagreement committee provides better representation than the consensus committee.

In fact, in the context of committees voting on multiple issues, a PR committee is guaranteed to make decisions that are perfectly consistent with majority preferences as long as all voters represented by a given candidate fully agree with that candidate on all issues. That is, under these assumptions the multiwinner variant of the Ostrogorski paradox is ruled out.

Proposition 1. Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$, such that k divides n . Consider also a set of candidates $W = \{w_1, \dots, w_k\}$ that provides PR for $(\mathcal{A}, k)$, and a partition of N into k pairwise disjoint subsets $N_1, \dots, N_k$ of size $\frac{n}{k}$ each such that for each $\ell \in [k]$ all voters in N_ℓ approve w_ℓ . Suppose that the elected committee W has to make decisions on a number of ‘yes/no’ issues by a majority vote (breaking ties in favor of ‘no’), and that for each issue and for each committee member $w_\ell \in W$ the opinion of all voters in N_ℓ coincides with the opinion of w_ℓ . Then, on each issue, the committee’s decision agrees with the opinion of the majority of the voters.

Proof. Fix an issue x . Let $W' \subseteq W$ be the subset of committee members that support x , and let $k' = |W'|$. Then $W \setminus W'$ is the subset of committee members that oppose x , and $|W \setminus W'| = k - k'$. A voter i supports x if and only if $i \in N_\ell$ for some $w_\ell \in W'$. It follows that x is supported by $\frac{n}{k} \cdot k'$ voters and opposed by $\frac{n}{k} \cdot (k - k')$ voters. It remains to observe that $k' > k - k'$ if and only if $\frac{n}{k} \cdot k' > \frac{n}{k} \cdot (k - k')$. Therefore, the committee decisions will always agree with the majority of the voters. $\square$

In contrast, if k divides n , but the committee does not provide PR,⁵ some voters would be overrepresented, while others would be underrepresented, and/or some voters would remain unrepresented. Therefore, the committee’s opinion may fail to follow that of the voters, even if each candidate agrees on all issues with all voters who approve her.

Example 3. Let $C = \{c_1, \dots, c_5\}$, and consider a ballot profile $\mathcal{A} = (A_1, \dots, A_9)$, where $A_i = \{c_1\}$ for $i = 1, \dots, 3$, $A_i = \{c_2\}$ for $i = 4, \dots, 6$, $A_7 = \{c_3\}$, $A_8 = \{c_4\}$, and $A_9 = \{c_5\}$ (see [Fig. 4](#)). The target committee size is $k = 3$.

The winning committee has to decide on three ‘yes/no’ issues. On each issue, the candidates follow the opinion of the voters that approve them. The opinions of the voters are as indicated in [Table 2](#).

Note that on each issue the opinion of the majority of the voters is ‘yes’. There are three committees that provide EJR, all of them composed of candidates c_1 and c_2 , plus one of c_3 , c_4 , or c_5 . However, each of these committees votes ‘no’ on one of the three issues.

⁵ In [Section 6](#), we will extend the notion of perfect representation to elections in which k does not divide n ; our discussion here can be easily adapted to this extended notion.

c_1			c_2			c_3	c_4	c_5
A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9

Fig. 4. Ballot profile in Example 3.

Now, in a real election, we cannot assume that the opinions of the voters represented by each candidate are unanimous on each issue, nor that each candidate will always follow the opinion of the voters that she represents. Therefore, we cannot guarantee that the decisions of the committee will always follow the opinion of the majority of the voters, even if we select a committee with desirable axiomatic properties. However, we believe that the core of our argument still holds: if a committee provides PR, we can assign voters to each candidate in such a way that each candidate represents the same number of voters and each voter approves her assigned candidate, whereas if the committee does not provide PR, voters will not be represented equitably. Therefore, if the only available information is the approvals of the voters, we believe that, in the context of voting on issues, the best possible committees are those that provide PR (whenever such committees exist). As a consequence, in this setting we may have to discard EJR due to its incompatibility with PR. Indeed, even if PR and EJR can be achieved simultaneously, there is no guarantee that a committee that provides both PR and EJR represents the voters better than a committee that only provides PR, as Example 2 shows.

We note that in real (political) elections, electing a voting body that will vote on issues in a way that agrees with voters' majority preferences is an important desideratum, but not the only one: it is also desirable to ensure that large cohesive groups are well represented. In such cases, one may need to achieve both PR and EJR, or, if this is not feasible, investigate tradeoffs between these criteria. One such tradeoff would be to consider a weakening of EJR that is compatible with PR; this is the route that we will explore in Section 4.

3.1.2. Participatory budgeting

In participatory budgeting, candidates are projects that can be implemented in a given city, each voter approves some of the projects, and the goal is to select a subset of projects to implement given a budget constraint [67]. The special case of this setting where all projects have the same cost (say, p) is equivalent to multi-winner voting: if the budget B satisfies $\lfloor \frac{B}{p} \rfloor = k$, then exactly k projects must be selected.

The crucial difference between selecting a committee to vote on issues and deciding on projects to implement is that, in the latter case, voters derive direct utility from each candidate in the winning committee they approve, whereas in the former case the representation quality offered by a committee is measured in terms of the outcomes of votes on issues. Thus, in the context of participatory budgeting it is natural to evaluate outcomes in terms of the individual satisfaction obtained by each voter. In particular, a voter who approves projects c_1 , c_2 , and c_3 clearly prefers an outcome W with $W \cap \{c_1, c_2, c_3\} = \{c_1, c_2\}$ to an outcome W' with $W' \cap \{c_1, c_2, c_3\} = \{c_1\}$ (it is less obvious that this voter prefers W'' with $W'' \cap \{c_1, c_2, c_3\} = \{c_2, c_3\}$ over W' , but if the only information available is the voters' approvals, this is perhaps a reasonable assumption).

Consequently, when we interpret Example 1 in the context of participatory budgeting, we believe that the consensus committee should be preferred over the disagreement committee. Thus, arguably, in this setting PR is perhaps less important, while EJR and similar axioms should play a more prominent role.

4. Proportional justified representation

The EJR axiom provides the following guarantee: at least one member of an ℓ -cohesive group has at least ℓ representatives in the committee. This focus on a single group member does not quite reflect our intuition of what it means for a group to be well-represented. A weaker and perhaps more natural condition is to require that collectively the members of an ℓ -cohesive group are allocated at least ℓ representatives. This idea is captured by the following definition.

Definition 3. Proportional justified representation (PJR) Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$. Given a positive integer $\ell \in [k]$, we say that a set of candidates W , $|W| = k$, provides ℓ -proportional justified representation (ℓ -PJR) for $(\mathcal{A}, k)$ if for every ℓ -cohesive set of voters $N^* \subseteq N$ it holds that $|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$. We say that W provides proportional justified representation (PJR) for $(\mathcal{A}, k)$ if it provides ℓ -PJR for all $\ell \in [k]$. An approval-based voting rule satisfies proportional justified representation (PJR) if for every ballot profile $\mathcal{A}$ and every target committee size k it outputs a committee that provides PJR for $(\mathcal{A}, k)$.

It is immediate that every committee that provides PJR also provides JR: the PJR condition for $\ell = 1$ is exactly JR. Also, it is easy to see that every committee that provides EJR also provides PJR: the condition " $|A_j \cap W| \geq \ell$ for some $j \in N^*$ " in the definition of EJR implies the condition " $|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$ " in the definition of PJR. To summarize, we obtain the following proposition.

Proposition 2. EJR implies PJR, and PJR implies JR.

Moreover, unlike EJR, PJR is compatible with PR. In fact, an even stronger statement is true: if a committee provides PR, it also provides PJR.

Theorem 3. For every profile $\mathcal{A} = (A_1, \dots, A_n)$ and every target committee size k , if a set of candidates W , $|W| = k$, provides PR, then W also provides PJR.

Proof. Let $W = \{w_1, \dots, w_k\}$. The assumption that W provides PR implies that k divides n . As W provides PR, there exist k pairwise disjoint subsets $N_1, \dots, N_k$ of size $\frac{n}{k}$ each such that for each $i \in [k]$ it holds that all voters in N_i approve w_i . Consider a set of agents $N^* \subseteq N$ and a positive integer ℓ such that $|N^*| \geq \ell \cdot \frac{n}{k}$. By the pigeonhole principle, N^* has a non-empty intersection with at least ℓ of the sets $N_1, \dots, N_k$. As each voter in $N^* \cap N_i$ approves w_i , it follows that the number of candidates in W that receive approvals from voters in N^* is at least ℓ . $\square$

At first glance, it may appear that a committee that provides PJR can leave many members of a large ℓ -cohesive group unrepresented: indeed, the definition of PJR will be satisfied as long as ℓ voters approve ℓ distinct candidates in the winning committee. However, our next result shows that, as long as a committee provides JR, the number of unrepresented voters cannot exceed $\lceil \frac{n}{k} \rceil - 1$ (and this bound is tight even if the committee in question provides EJER).

Proposition 3. Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$. Fix a size- k committee W that provides JR for $(\mathcal{A}, k)$ and a 1-cohesive group $N^* \subseteq N$. Then there are at most $\lceil \frac{n}{k} \rceil - 1$ voters in N^* who are not represented by W .

Proof. Let $N' \subseteq N^*$ be the subset of voters in N^* who are not represented by W . Since N^* is a 1-cohesive group, we have $\cap_{i \in N^*} A_i \neq \emptyset$ and hence $\cap_{i \in N'} A_i \neq \emptyset$. Suppose for the sake of contradiction that $|N'| > \lceil \frac{n}{k} \rceil - 1$. This implies that $|N'| \geq \frac{n}{k}$, i.e., the voters in N' form a 1-cohesive group themselves. Hence the JR axiom requires that at least one of them approves at least one candidate in W , a contradiction with the choice of N' . $\square$

Example 4. Let k and p be positive integers with $k, p \geq 2$. Consider the set of candidates $C = A \cup B \cup C'$, where A , B and C' are pairwise disjoint, $|A| = |B| = k - 1$, and $C' = \{c^*\}$ is a singleton. Consider also a set of voters $N = N_1 \cup N_2 \cup N_3$ with a ballot profile $(A_i)_{i \in N}$, where

- $|N_1| = p - 1$ and $A_i = A$ for each $i \in N_1$,
- $|N_2| = (k - 2)p + 1$ and $A_i = A \cup B$ for each $i \in N_2$, and
- $|N_3| = p$ and $A_i = \{c^*\}$ for each $i \in N_3$.

For this election we have

$$\frac{n}{k} = \frac{|N_1| + |N_2| + |N_3|}{k} = \frac{kp}{k} = p, \text{ and hence } \lceil \frac{n}{k} \rceil = p = |N_1| + 1.$$

Observe that the committee $W = B \cup \{c^*\}$ provides EJER for this election and target committee size k . Indeed, consider an ℓ -cohesive group N^* for some $\ell \in [k]$. Let z be some candidate approved by all voters in N^* . If $z = c^*$ then $N^* \subseteq N_3$ and hence $\ell = 1$; then N^* is represented in W by c^* . Otherwise we have $z \in A \cup B$ and hence $N^* \subseteq N_1 \cup N_2$; note that this implies that $|N^*| \leq |N_1| + |N_2| = (k - 1)p$ and hence $\ell \leq k - 1$. Observe also that $|N^*| \geq \frac{n}{k} = p$ and hence $N^* \cap N_2 \neq \emptyset$. As every voter in $N^* \cap N_2$ approves all candidates in B , $|B| = k - 1 \geq \ell$ and $B \subseteq W$, the EJER condition is satisfied in this case as well.

However, the $p - 1 = \lceil \frac{n}{k} \rceil - 1$ voters in N_1 remain unrepresented in W , so the bound in Proposition 3 is tight.

We will now argue that a committee that provides PJR can be computed in polynomial time as long as the target committee size k divides the number of voters n . Specifically, we will show that under this condition, both the Monroe rule and Greedy Monroe (the latter of which is polynomial-time computable) satisfy PJR. We note that PAV satisfies EJER and hence PJR even if k does not divide n ; however, computing the output of PAV is NP-hard.

Theorem 4. Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$. If the target committee size k divides n then the outputs of Monroe and Greedy Monroe on $(\mathcal{A}, k)$ satisfy PJR.

Proof.

Let $s = \frac{n}{k}$; note that $s \in \mathbb{N}$. Suppose that the Monroe rule outputs a committee W ; assume without loss of generality that $W = \{c_1, \dots, c_k\}$. Let π be a valid mapping $N \rightarrow W$ whose Monroe score is equal to the Monroe score of W . Let $N_j = \pi^{-1}(c_j)$ for $j = 1, \dots, k$. Suppose for the sake of contradiction that there exists an $\ell > 0$ such that ℓ -PJR is violated for some ℓ -cohesive group of voters N^* . Note that this means that at least one of the ℓ candidates jointly approved by voters in N^* does not appear in W ; let c be some such candidate.

Let us say that a set N_j is *good* if at least one voter in $N_j \cap N^*$ approves c_j , and *bad* otherwise. Note that if there are at least ℓ good sets among $N_1, \dots, N_k$ then there are ℓ distinct candidates in W each of which is approved by at least one voter in N^* , a contradiction with our choice of N^* .

Thus, there are at most $\ell - 1$ good sets, so at most $(\ell - 1) \cdot s$ voters from N^* appear in good sets. As $|N^*| \geq \ell \cdot s$, there are at least s voters in N^* who appear in bad sets. These voters are assigned by π to candidates they do not approve; let $N' \subseteq N^*$ be a set that consists of s such voters.

Let N_j be some bad set with $N_j \cap N' \neq \emptyset$ and let $W' = (W \setminus \{c_j\}) \cup \{c\}$. We now construct a valid mapping $\pi' : N \rightarrow W'$ as follows. We let $\pi'(i) = \pi(i)$ for all $i \in N \setminus (N' \cup N_j)$. Then, we set $\pi'(i) = c$ for each $i \in N'$. At this point, the voters in $N_j \setminus N'$ are not yet assigned by π' , and, on the other hand, each candidate $c_\ell \in W \setminus \{c_j\}$ only receives $|N_\ell \setminus N'|$ votes, i.e., they ‘miss’ $|N_\ell \cap N'|$ votes; we need to define π' on $N_j \setminus N'$ to ‘compensate’ these candidates. Note that $N' = \cup_{\ell \in [k]} (N_\ell \cap N')$ and hence the total number of ‘missing’ votes is

$$\sum_{c_\ell \in W \setminus \{c_j\}} |N_\ell \cap N'| = |N'| - |N_j \cap N'| = |N' \setminus N_j|.$$

Proof. Aziz et al. [1] show that if $\mathbf{w} \neq (1, 0, \dots)$ then Seq- $\mathbf{w}$ -Thiele fails JR; it follows that it also fails PJR.

To show that Seq- $\mathbf{w}$ -Thiele fails PJR for $\mathbf{w} = (1, 0, \dots, 0)$, consider the following election. Let $n = 6$, $k = 3$, $C = \{c_1, c_2, c_3, c_4\}$. Set $A_1 = \dots = A_4 = \{c_1, c_2\}$, $A_5 = \{c_3\}$, $A_6 = \{c_4\}$. Seq- $(1, 0, \dots)$ -Thiele outputs a committee that contains c_3 , c_4 and exactly one of the remaining candidates. However, the set $\{1, 2, 3, 4\}$ is 2-cohesive, and thus PJR requires that it gets two representatives. $\square$

5. Average satisfaction

A useful measure in the context of fair representation is that of *average satisfaction*: given a ballot profile $(A_1, \dots, A_n)$, a committee W , and a group of voters $N^* \subseteq N$, we define the *average satisfaction* of the voters in N^* as

$$\frac{1}{|N^*|} \sum_{i \in N^*} |A_i \cap W|. \quad (1)$$

While it is not always possible to ensure that every group of voters has high average satisfaction, it is natural to ask if we can provide some guarantees with respect to this measure to groups of voters that are large and cohesive.

Our first observation is that if a committee W provides JR, we can derive a lower bound on the average satisfaction of such groups.⁷

Proposition 6. Consider a ballot profile $(A_1, \dots, A_n)$, and suppose that the target committee size k divides n . Let W be a committee of size k that provides JR. Then for every $\ell \in [k]$ and for every ℓ -cohesive group of voters N^* we have

$$\frac{1}{|N^*|} \sum_{i \in N^*} |A_i \cap W| \geq 1 - \frac{1}{\ell} + \frac{k}{\ell n}.$$

Proof. Let $s = \frac{n}{k}$. Fix an $\ell \in [k]$ and an ℓ -cohesive group of voters N^* . Since W provides JR, Proposition 3 implies that at most $s - 1$ voters in N^* are unrepresented in W . Hence,

$$\frac{1}{|N^*|} \sum_{i \in N^*} |A_i \cap W| \geq \frac{|N^*| - s + 1}{|N^*|} = 1 - \frac{s - 1}{|N^*|} \geq 1 - \frac{s - 1}{\ell s} = 1 - \frac{1}{\ell} + \frac{k}{\ell n}.$$

$\square$

However, for voting rules that satisfy EJR we can obtain a much stronger guarantee.

Theorem 5. Consider a ballot profile $(A_1, \dots, A_n)$, and suppose that the target committee size k divides n . Let W be a committee of size k that provides EJR. Then for every $\ell \in [k]$ and for every ℓ -cohesive group of voters $N^* \subseteq N$ we have

$$\frac{1}{|N^*|} \sum_{i \in N^*} |A_i \cap W| \geq \frac{\ell - 1}{2}.$$

Proof. Let $s = \frac{n}{k}$. Fix an $\ell \in [k]$ and an ℓ -cohesive group of voters N^* ; let $n^* = |N^*|$. For each $j \in [\ell]$, we say that a voter in N^* is *j-happy* if she approves at least j candidates in W , and *j-unhappy* otherwise.

We claim that for each $j \in [\ell]$ the set N^* contains at most $j \cdot s - 1$ voters who are *j-unhappy*. Indeed, suppose that for some $j \in [\ell]$ there are at least $j \cdot s$ voters in N^* who are *j-unhappy*. As all these voters are in N^* , there is a set of $\ell \geq j$ candidates that they all approve; hence, these voters form a j -cohesive group, yet none of them is *j-happy*, a contradiction with the assumption that W provides EJR.

It follows that N^* contains at least $n^* - \ell \cdot s + 1$ voters who are ℓ -happy. Let N_ℓ be a set that consists of exactly $n^* - \ell \cdot s + 1$ voters who are ℓ -happy.

Similarly, N^* contains at least $n^* - (\ell - 1) \cdot s + 1$ voters who are $(\ell - 1)$ -happy. Thus, $N^* \setminus N_\ell$ contains at least $(n^* - (\ell - 1) \cdot s + 1) - (n^* - \ell \cdot s + 1) = s$ voters who are $(\ell - 1)$ -happy; let $N_{\ell-1}$ denote a set that consists of exactly s voters in $N^* \setminus N_\ell$ who are $(\ell - 1)$ -happy.

We continue inductively in the same manner: for each $j = \ell - 2, \dots, 1$ we construct a set N_j of exactly s voters who are *j-happy* so that the sets $N_1, \dots, N_\ell$ are all pairwise disjoint. This is possible exactly because for each $j = \ell - 2, \dots, 1$ there are at least $n^* - j \cdot s + 1$ voters who are *j-happy*, while

$$|N_\ell| + |N_{\ell-1}| + \dots + |N_{j+1}| = n^* - \ell \cdot s + 1 + (\ell - 1 - j) \cdot s = n^* - j \cdot s + 1 - s.$$

The average satisfaction of voters in $N^* \setminus N_\ell$ is then at least

$$\frac{1}{|N^* \setminus N_\ell|} \sum_{i \in N^* \setminus N_\ell} |A_i \cap W| \geq \frac{1}{\ell s - 1} \cdot \sum_{j=1}^{\ell-1} |N_j| \cdot j \geq \frac{1}{\ell s} \cdot \frac{s(\ell-1)\ell}{2} = \frac{\ell-1}{2},$$

whereas the average satisfaction of the voters in N_ℓ is at least ℓ . As the average satisfaction of voters in N^* is a convex combination of these two quantities, it is at least $\frac{\ell-1}{2}$. $\square$

⁷ Throughout this section, we assume that k divides n . This assumption allows us to avoid technical complications, but similar (albeit slightly weaker) results can be obtained if k is not required to divide n .

In contrast, the worst-case guarantee provided by PJR is not any stronger than the one provided by JR alone.

Example 6. Consider a ballot profile $(A_1, \dots, A_n)$ over a candidate set $C = \{c_1, \dots, c_n, d_1, \dots, d_n\}$ where $A_i = \{c_1, \dots, c_n, d_i\}$ for $i \in [n]$. For $k = n$, the committee $\{d_1, \dots, d_n\}$ provides PJR (and PR), but the average satisfaction of the voters in N (which form an n -cohesive group) is only 1.

Remark 3. We say that a group of voters N^* is *very cohesive* if all voters in N^* approve exactly the same set of candidates. For very cohesive groups, the average satisfaction guarantees provided by PJR are as high as those provided by EJR, and much higher than those provided by JR. Formally, consider an election in which there is a group of voters N^* , $|N^*| \geq \ell \cdot \frac{n}{k}$, who all approve precisely the same set of candidates S , $|S| \geq \ell$. Then every committee that provides PJR (or EJR) for this election elects at least ℓ members of S , thereby ensuring that the average satisfaction of voters in N^* is at least ℓ . In contrast, a committee that provides JR may select just one member of S , in which case the average satisfaction of voters in N^* will be just 1.

6. Fractional perfect representation

As discussed in Section 3, the perfect representation axiom has an important limitation: it only constrains the behavior of a voting rule in settings where the committee size k divides the number of ballots n . We now propose a more general version of this axiom, which applies to all values of k and n .

Definition 4. Fractional perfect representation (FPR) Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$. We say that a set of candidates W , $|W| = k$, provides *fractional perfect representation (FPR)* for $(\mathcal{A}, k)$ if there exists a collection of non-negative reals $\mathcal{V} = (v_{i,w})_{i \in N, w \in W}$ satisfying

$$\begin{aligned} 0 \leq v_{i,w} &\leq 1 \quad \text{for all } i \in N \text{ and } w \in W, \\ v_{i,w} > 0 &\Rightarrow w \in A_i \quad \text{for all } i \in N \text{ and } w \in W, \\ \sum_{w \in W} v_{i,w} &= 1 \quad \text{for all } i \in N, \text{ and} \\ \sum_{i \in N} v_{i,w} &= \frac{n}{k} \quad \text{for all } w \in W. \end{aligned}$$

An approval-based voting rule *satisfies FPR* if for every profile $\mathcal{A}$ and every target committee size k , the rule outputs a committee that provides FPR for $(\mathcal{A}, k)$ whenever such a committee exists.

The intuition behind FPR is that, like PR, it requires that each member of W represents exactly $\frac{n}{k}$ voters, but it relaxes the way a voter can be represented in the committee, by allowing fractional representation.

We remark that not every instance of the committee selection problem admits a committee that provides FPR. Consider, e.g., the ballot profile $\mathcal{A} = (A_1, A_2)$ with $A_1 = \{a\}$ and $A_2 = \{b\}$. For $k = 1$, one of the two voters remains unrepresented, no matter which W we select. Hence, for every choice of $W \in \{\{a\}, \{b\}\}$ it is impossible to satisfy the condition $\sum_{i \in N} v_{i,w} = \frac{n}{k} = 2$ for $w \in W$. On such instances, the FPR axiom is not binding.

It is convenient to think of the FPR axiom in terms of network flows. Specifically, given a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C and a committee $W \subseteq C$, $|W| = k$, we construct a flow network $\mathcal{N}_{\mathcal{A},W}$ with source s , sink t , the set of nodes $N \cup W \cup \{s, t\}$, and the following set of arcs: (1) for each $i \in N$, a unit-capacity arc from s to i ; (2) for each $i \in N$ and each $w \in A_i$, a unit-capacity arc from i to w ; (3) for each $w \in W$, an arc of capacity $\frac{n}{k}$ from w to t . Then there is a one-to-one correspondence between collections of reals $(v_{i,w})_{i \in N, w \in W}$ that satisfy the conditions in Definition 4 and flows of size n in this network: the quantity $v_{i,w}$ can be interpreted as the flow on the arc (i, w) and vice versa.

This connection to network flows enables us to establish a relationship between PR and FPR for the case where k divides n .

Theorem 6. For every profile $\mathcal{A} = (A_1, \dots, A_n)$ and every target committee size k such that k divides n it holds that a size- k subset of candidates provides FPR if and only if it provides PR.

Proof. It is immediate that PR implies FPR. Indeed, suppose that a committee W provides PR for $(\mathcal{A}, k)$. We can assume without loss of generality that $W = \{c_1, \dots, c_k\}$, and that the voters are partitioned into k pairwise disjoint subsets $N_1, \dots, N_k$ of size $\frac{n}{k}$ each so that for each $\ell \in [k]$ all voters in N_ℓ approve c_ℓ . We can then set $v_{i,w} = 1$ if there exists an $\ell \in [k]$ such that $i \in N_\ell$, $w = c_\ell$ and $v_{i,w} = 0$ otherwise. Then the collection $\mathcal{V} = (v_{i,w})_{i \in N, w \in W}$ witnesses that W provides FPR for $(\mathcal{A}, k)$.

To see that FPR implies PR when k divides n , suppose that a committee W provides FPR for $(\mathcal{A}, k)$. Then the network $\mathcal{N}_{\mathcal{A},W}$ admits a flow of size n . Crucially, if k divides n , all arcs of this network have integer capacity, and hence the network admits an integer flow of size n . Fix some such flow. For each $i \in N$, there is exactly one arc leaving i that carries one unit of flow, and for each $w \in W$ there are exactly $\frac{n}{k}$ flow-carrying arcs entering w . Hence, any integer flow in $\mathcal{N}_{\mathcal{A},W}$ serves as a witness that W provides PR for $(\mathcal{A}, k)$. $\square$

Theorem 6 indicates that FPR is a generalization of PR. The following example shows that it is strictly more demanding.

Example 7. Let $C = \{a, b, c\}$, and consider the ballot profile $\mathcal{A} = (A_1, A_2, A_3)$, where $A_1 = \{a, c\}$, $A_2 = \{b, c\}$, and $A_3 = \{b\}$ (see Fig. 6). Set $k = 2$. Note that k does not divide n , so the PR axiom does not apply. However, the FPR axiom is binding. Indeed, consider the committee $W = \{b, c\}$. Observe that W provides FPR: we can set $v_{1c} = 1$, $v_{2b} = v_{2c} = .5$, $v_{3b} = 1$. On the other hand, $W' = \{a, c\}$ and $W'' = \{a, b\}$ do not provide FPR: indeed, both W' and W'' contain a , but there is only one voter who approves a , so the condition $v_{1a} + v_{2a} + v_{3a} = 1.5$ cannot be satisfied. Thus, any rule that satisfies FPR must output W on $(\mathcal{A}, k)$.

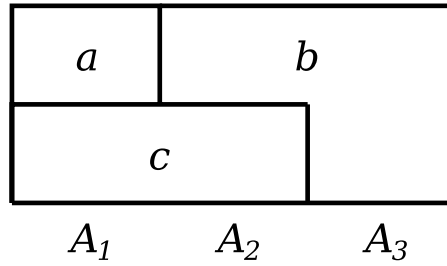


Fig. 6. Ballot profile in Example 7.

Theorem 6 also implies that deciding whether a committee provides FPR is NP-complete.

Proposition 7. Given a ballot profile $\mathcal{A}$ and a target committee size k , it is NP-complete to decide whether there exists a committee that provides FPR for $(\mathcal{A}, k)$.

Proof. NP-hardness follows from Theorems 1 and 6. To prove that this problem is in NP, we observe that, given an n -voter ballot profile $\mathcal{A}$, a target committee size k , and a set of candidates W of size k , we can construct the network $\mathcal{N}_{\mathcal{A}, W}$ and check if it admits a flow of size n ; the latter problem is known to be polynomial-time solvable. $\square$

6.1. FPR and the monroe rule

We have argued that the Monroe rule satisfies PR. It is then natural to ask whether it satisfies FPR. It turns out that the answer depends on how we break ties. We first observe that Example 7 can be used to show that the Monroe rule may output a committee that does not provide FPR.

Proposition 8. There exists a ballot profile $\mathcal{A}$ and a target committee size k such that there exists a committee W , $|W| = k$, that provides FPR for $(\mathcal{A}, k)$, but the Monroe rule may output a set of candidates W' that does not provide FPR for $(\mathcal{A}, k)$.

Proof. Consider the profile $\mathcal{A}$ in Example 7 and committees $W = \{b, c\}$ and $W' = \{a, b\}$. We have argued that W provides FPR, but W' does not. However, both committees have a Monroe score of 3, so the Monroe rule may output W' . $\square$

On the other hand, a committee that provides FPR always has the maximum possible Monroe score.

Proposition 9. Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a committee $W \subseteq C$, $|W| = k$. If W provides FPR for $(\mathcal{A}, k)$, then the Monroe score of W is n .

Proof. Consider again the flow network $\mathcal{N}_{\mathcal{A}, W}$. We turn it into an instance of the circulation problem with capacities and lower bounds as follows. For each arc leaving s we set both its capacity and lower bound to 1. For each arc between N and W we set its capacity to 1 and the lower bound to 0. For each arc entering t we set its capacity to $\lceil \frac{n}{k} \rceil$ and lower bound to $\lfloor \frac{n}{k} \rfloor$. We also add an arc from t to s and set its capacity and lower bound to n .

If W provides FPR for $(\mathcal{A}, k)$, the collection $\mathcal{V} = (v_{i,w})_{i \in N, w \in W}$ corresponds to a valid circulation in the modified network: in this circulation, the flow on each arc leaving s is 1, the flow on each arc entering t is $\frac{n}{k}$, the flow on the $t-s$ arc is n and the flow on each arc of the form (i, w) is $v_{i,w}$. On the other hand, in the modified network all capacities and lower bounds are integers and hence it admits an integer circulation. Any such circulation corresponds to a valid mapping $\pi : N \rightarrow W$, and the Monroe score of this mapping is n . $\square$

Together, Propositions 8 and 9 imply the following corollary.

Corollary 1. The Monroe rule satisfies FPR as long as ties are broken in favor of committees that provide FPR.

We note that there exist voting rules that satisfy FPR irrespective of the tie-breaking rule. Indeed, Brill et al. [68] prove that two optimization variants of the Phragmén rule (namely, leximax-Phragmén and var-Phragmén) satisfy PR, and their proof can be used to show that these rules also satisfy FPR. Moreover, leximax-Phragmén satisfies PJR [68] and even the stronger property known as priceability [20] (discussed later in this section). To the best of our knowledge, leximax-Phragmén is the only known rule satisfying both priceability and FPR. For readability, we omit the definitions of these rules and the formal proofs that they satisfy FPR.

6.2. FPR and other proportionality axioms

Recall that in Section 3 we have observed that PR is incompatible with EJR; Theorem 6 implies that the incompatibility of PR and EJR (as well as PR and the core) is inherited by FPR. This incompatibility motivated us to introduce the PJR axiom. It is therefore natural to ask if the FPR axiom is compatible with PJR. It turns out that the answer is ‘yes’. In fact, a stronger claim is true: every committee that provides FPR also provides PJR.

Theorem 7. For every profile $\mathcal{A} = (A_1, \dots, A_n)$ and every target committee size k , if a set of candidates W , $|W| = k$, provides FPR for $(\mathcal{A}, k)$, then W also provides PJR.

Proof. Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$, a target committee size k , a set of candidates W that provides FPR for $(\mathcal{A}, k)$, and a flow of size n in the network $\mathcal{N}_{\mathcal{A}, W}$. Let N^* be an ℓ -cohesive group of voters. Observe that in $\mathcal{N}_{\mathcal{A}, W}$ all flow leaving N^* must arrive to $W \cap (\bigcup_{i \in N^*} A_i)$. There are $|N^*| \geq \ell \cdot \frac{n}{k}$ units of flow leaving N^* towards W and $|W \cap (\bigcup_{i \in N^*} A_i)| \cdot \frac{n}{k}$ units of flow leaving $W \cap (\bigcup_{i \in N^*} A_i)$ towards t . Hence, $|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$. $\square$

Consider a committee W that provides FPR. By Theorem 7 it also provides PJR. Hence, Proposition 6 implies a lower bound on the average satisfaction guaranteed by W . However, one can get a stronger lower bound directly from the definition of FPR: FPR implies that each voter has at least one representative in W , so the average satisfaction guaranteed by W is at least 1. In contrast, for very cohesive groups of voters the connection to PJR is useful: it allows us to obtain even stronger average satisfaction guarantees by Remark 3.

Proposition 10. If a committee provides FPR, the average satisfaction of any group of voters is at least 1. Moreover, consider a group of voters N^* such that $|N^*| \geq \ell \cdot \frac{n}{k}$ and for all $i \in N^*$ it holds that $A_i = S$ for some $S \subseteq C$ with $|S| \geq \ell$. Then the average satisfaction of N^* is at least ℓ .

The lower bound established in Proposition 10 is tight: Example 6 shows that it is not possible to achieve stronger worst-case guarantees for committees that provide FPR (or PR).

FPR is also compatible with a somewhat stronger version of PJR, recently proposed by Aziz and Lee [21], called Inclusion PSC (IPSC). IPSC is defined for weak preferences. In a recent paper, Brill and Peters [25] have renamed the specialization of IPSC to approval-based elections as PJR+. This axiom is formulated as follows.

Definition 5. PJR+ Consider a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C , and a target committee size k , $k \leq |C|$. A set of candidates W , $|W| = k$, provides PJR+ (IPSC for approval-based elections) if there does not exist a positive integer $\ell \in [k]$ and a set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \cdot \frac{n}{k}$ such that $|W \cap \bigcup_{i \in N^*} A_i| < \ell$, but $(\bigcap_{i \in N^*} A_i) \setminus W \neq \emptyset$.

It is not hard to check that PJR+ implies PJR. Indeed, if a committee W does not provide PJR, there exists an $\ell \in [k]$ and an ℓ -cohesive group of voters N^* such that $|W \cap \bigcup_{i \in N^*} A_i| < \ell$. But then, since $|\bigcap_{i \in N^*} A_i| \geq \ell$, it follows that $(\bigcap_{i \in N^*} A_i) \setminus W \neq \emptyset$, i.e., W does not provide PJR+ either. Moreover, PJR+ is more demanding than PJR in that it offers non-trivial representation guarantees to large groups even if they only agree on a single candidate; see the work of Brill and Peters [25] for specific examples.

We will now argue that FPR implies PJR+.

Proposition 11. For every profile $\mathcal{A} = (A_1, \dots, A_n)$ and every target committee size k , if a size- k set of candidates W provides FPR for $(\mathcal{A}, k)$, then W also provides PJR+.

Proof. In the proof of Theorem 7, we argue that if W is a size- k committee that provides FPR and N^* is an ℓ -cohesive group of voters, then $|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$. In fact, for this argument to go through, it suffices to assume that $|N^*| \geq \ell \cdot \frac{n}{k}$. From this, the proposition follows. $\square$

We will now discuss the relationship between FPR and two other proportionality axioms that have been recently been put forward by Peters and Skowron [20]: priceability and laminar proportionality.

Definition 6. Priceability (Peters and Skowron [20]) A price system for a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ over a candidate set C is a pair $(p, \mathbf{q})$, where $p > 0$ is the price, and $\mathbf{q} = (q_i)_{i \in N}$, where $q_i : C \rightarrow [0, 1]$, is the payment function of voter i , which satisfies the following conditions:

- for each $c \in C \setminus A_i$ it holds that $q_i(c) = 0$, and
- $\sum_{c \in C} q_i(c) \leq 1$.

A set of candidates W is priceable if there exists a price system $(p, \mathbf{q})$ satisfying

- $\sum_{i \in N} q_i(w) = p$ for each $w \in W$,
- $\sum_{i \in N} q_i(c) = 0$ for each $c \in C \setminus W$, and
- $\sum_{i \in N : c \in A_i} (1 - \sum_{w \in W} q_i(w)) \leq p$ for each $c \in C \setminus W$.

The next proposition establishes a tight relationship between FPR and priceability with price $p = \frac{n}{k}$.

Proposition 12. For every ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ and every target committee size k , a size- k set of candidates W provides FPR for $(\mathcal{A}, k)$ if and only if W is priceable with price $p = \frac{n}{k}$.

Proof. Suppose first that W provides FPR for $(\mathcal{A}, k)$, and let $(v_{i,w})_{i \in N, w \in W}$ be the respective collection of non-negative reals. We can then define the functions $(q_i)_{i \in N}$ by setting $q_i(w) = v_{i,w}$ for $w \in W$, $q_i(c) = 0$ for $c \in C \setminus W$. Then the price system $(\frac{n}{k}, (q_i)_{i \in N})$ witnesses that W is priceable; in particular, for each $c \in C \setminus W$ we have $\sum_{i \in N : c \in A_i} (1 - \sum_{w \in W} q_i(w)) = 0$.

Conversely, suppose that the set W is priceable, as witnessed by a price system $(\frac{n}{k}, (q_i)_{i \in N})$. Set $v_{i,w} = q_i(w)$ for all $w \in W$. Then, by construction, for all $i \in N$, $w \in W$ we have $0 \leq v_{i,w} \leq 1$ and $v_{i,w} = 0$ whenever $w \notin A_i$. Moreover, $\sum_{i \in N} v_{i,w} = \frac{n}{k}$ for each $w \in W$. It then follows that $\sum_{w \in W} \sum_{i \in N} v_{i,w} = n$; as $\sum_{w \in W} v_{i,w} = \sum_{c \in C} q_i(c) \leq 1$ for all $i \in N$, it has to be the case that $\sum_{w \in W} v_{i,w} = 1$ for each $i \in N$. Hence, $(v_{i,w})_{i \in N, w \in W}$ witnesses that W provides FPR. $\square$

Peters and Skowron [20] prove that if an approval-based multi-winner voting rule always outputs priceable committees, then this rule is an extension of the D'Hondt apportionment method (see, e.g., the book of Pukelsheim [28] for a survey of apportionment methods). In contrast, later in this section we will show that, in the context of apportionment, FPR reduces to a weaker axiom, which is called *weak proportionality* [[27], p. 96–97]. This axiom is satisfied by any reasonable apportionment method, including any divisor method and the largest remainders method. In particular, there exists a voting rule that satisfies FPR, JR, and is an extension of the Sainte-Laguë method⁸: specifically, the rule var-Phragmén [71–74] has all these properties (the survey by Janson [77] shows that var-Phragmén is an extension of the Sainte-Laguë method, and Brill et al. [68] prove that var-Phragmén satisfies both PR and JR; their proof for PR extends easily to FPR).

Another concept introduced by Peters and Skowron [20] is *laminar proportionality*. To define it, we need to introduce additional notation. Given a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$, denote the profile $(A_1 \setminus \{c\}, \dots, A_n \setminus \{c\})$ by $\mathcal{A}_{\setminus c}$.

Definition 7. Laminar proportionality (Peters and Skowron [20]) An election with a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ and target committee size k is *laminar* if one of the following conditions holds:

- (UP) (unanimous profile) $A_1 = A_2 = \dots = A_n$, and $|A_1| \geq k$,
- (UC) (unanimous candidate) $k \geq 2$ and there is a candidate $c \in C$ such that $c \in A_i$ for all $i \in [n]$ and $(\mathcal{A}_{\setminus c}, k-1)$ is laminar, but does not satisfy (UP), or
- (LD) (laminar decomposition) There exist two ballot profiles $\mathcal{A}^1 = (A_1^1, \dots, A_{n_1}^1)$ and $\mathcal{A}^2 = (A_1^2, \dots, A_{n_2}^2)$ and two target committee sizes k_1 and k_2 such that: (i) $\mathcal{A}$ is obtained by permuting the list $(A_1^1, \dots, A_{n_1}^1, A_1^2, \dots, A_{n_2}^2)$; (ii) $k = k_1 + k_2$; (iii) $(\mathcal{A}^1, k_1)$ and $(\mathcal{A}^2, k_2)$ are laminar; (iv) $\bigcup_{i \in [n_1]} A_i^1 \cap \bigcup_{i \in [n_2]} A_i^2 = \emptyset$; and (v) $\frac{n_1}{k_1} = \frac{n_2}{k_2}$.

Given a laminar election $(\mathcal{A}, k)$, we say that a set of candidates W of size k is *laminar proportional* if one of the following conditions holds:

1. $(\mathcal{A}, k)$ satisfies (UP) and $W \subseteq A_1$,
2. $(\mathcal{A}, k)$ satisfies (UC) with c as a unanimous candidate, $c \in W$, and $W \setminus \{c\}$ is laminar proportional for election $(\mathcal{A}_{\setminus c}, k-1)$, or
3. $(\mathcal{A}, k)$ satisfies (LD), and W can be decomposed as $W = W_1 \cup W_2$, where W_1 is laminar proportional for $(\mathcal{A}^1, k_1)$ and W_2 is laminar proportional for $(\mathcal{A}^2, k_2)$.

An approval-based multi-winner voting rule is *laminar proportional* if, given a laminar election $(\mathcal{A}, k)$, it outputs a laminar proportional set of candidates.

The relationship between FPR and laminar proportionality is given by the following theorem.

Theorem 8. Consider a laminar election with a ballot profile $\mathcal{A} = (A_1, \dots, A_n)$ and target committee size k . If a set of candidates W is laminar proportional for $(\mathcal{A}, k)$, then W also provides FPR for $(\mathcal{A}, k)$; however, the converse is not true.

Proof. If $(\mathcal{A}, k)$ satisfies (UP), we can simply set $v_{i,w} = 1/k$ for each voter $i \in N$ and each candidate $w \in W$. If $(\mathcal{A}, k)$ satisfies (UC) with $c \in W$ as a unanimous candidate, we recursively construct a collection $\mathcal{V}' = (v'_{i,w})_{i \in N, w \in W \setminus \{c\}}$ for election $(\mathcal{A}_{\setminus c}, k-1)$ and committee $W \setminus \{c\}$, set $v_{i,w} = v'_{i,w} \cdot \frac{k-1}{k}$ for each $i \in N$ and each candidate $w \in W \setminus \{c\}$, and then set $v_{i,c} = 1/k$ for each $i \in N$. Finally, if $(\mathcal{A}, k)$ satisfies (LD), we recursively construct a collection $\mathcal{V}' = (v'_{i,w})_{i \in N, w \in W_1}$ for election $(\mathcal{A}^1, k_1)$ and committee W_1 , and a collection $\mathcal{V}'' = (v''_{i,w})_{i \in N, w \in W_2}$ for election $(\mathcal{A}^2, k_2)$ and committee W_2 . Then for each $i \in N$, $w \in W$ we set

$$v_{i,w} = \begin{cases} v'_{i,w} & \text{if } i \in [n_1], w \in W_1 \\ v''_{i,w} & \text{if } i \in N \setminus [n_1], w \in W_2 \\ 0 & \text{otherwise.} \end{cases}$$

To see that the converse is not true, consider a profile $(\mathcal{A}, k)$ where $k = 4$, $\mathcal{A} = (A_1, A_2, A_3, A_4)$, $A_1 = A_2 = \{a, b, c, d\}$, and $A_3 = A_4 = \{a, b, c', d'\}$. Note that a set of candidates W of size 4 is laminar proportional for $(\mathcal{A}, k)$ if and only if $\{a, b\} \subset W$ and $|W \cap \{c, d\}| = |W \cap \{c', d'\}| = 1$. However, $W' = \{c, d, c', d'\}$ provides FPR for $(\mathcal{A}, k)$: indeed, we can set $v_{1,c} = v_{2,d} = v_{3,c'} = v_{4,d'} = 1$. $\square$

We conclude that FPR is incomparable with laminar proportionality. Indeed, it is weaker than laminar proportionality in the sense that for laminar elections there may exist committees that provide FPR, but not laminar proportionality (as shown in the second part of the proof of Theorem 8). On the other hand, it is stronger than laminar proportionality in the sense that it constrains the behavior of voting rules on a larger set of elections (e.g., the election in Example 7 is not laminar, but has an FPR committee).

6.3. FPR and individual representation axioms

Unlike JR, PJR, and EJR, the FPR axiom provides representation guarantees to individual voters rather than groups of voters. The literature on multi-winner approval voting proposes several other axioms of this type. This includes, in particular, semi-strong and

⁸ Since the Sainte-Laguë method does not satisfy lower quota, and EJR and PJR reduce to lower quota for apportionment methods [29], no extension of Sainte-Laguë can satisfy EJR or PJR.

strong justified representation, put forward by Aziz et al. [1], and individual representation, as defined by Brill et al. [78]. Semi-strong justified representation requires that for each 1-cohesive group it holds that each voter in this group approves some member of the winning committee. The strong justified representation axiom is even more demanding: it requires that for each 1-cohesive group the winning committee contains some candidate approved by all group members. Finally, individual representation requires that each member of each ℓ -cohesive group approves at least ℓ members of the winning committee.

It is immediate that FPR is stronger than semi-strong justified representation, in the sense that if a committee provides the former, it also provides the latter (this follows, e.g., from Proposition 10), but the converse is not true (this follows from part (3) of Proposition 13 below). In contrast, FPR can be shown to be incomparable with the other two axioms.

Proposition 13. *FPR is incomparable with strong justified representation and individual representation, in the following sense:*

- (1) *There exists an election $(\mathcal{A}, k)$ over a candidate set C and a committee $W \subseteq C$ with $|W| = k$ such that W provides FPR, but not strong justified representation.*
- (2) *There exists an election $(\mathcal{A}, k)$ over a candidate set C and a committee $W \subseteq C$ with $|W| = k$ such that W provides FPR, but not individual representation.*
- (3) *There exists an election $(\mathcal{A}, k)$ over a candidate set C and a committee $W \subseteq C$ with $|W| = k$ such that W provides strong justified representation and individual representation, but not FPR.*

Proof. We provide a separate construction for each claim.

- (1) Consider a ballot profile (A_1, A_2, A_3, A_4) with $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, $A_3 = \{c, d\}$, $A_4 = \{d, a\}$, and $k = 2$. The committee $\{a, c\}$ provides FPR for this election (we can set $v_{1,a} = v_{4,a} = 1$, $v_{2,c} = v_{3,c} = 1$). However, it does not provide strong justified representation: the first two voters form a 1-cohesive group and approve b , but $b \notin W$.
- (2) Consider a ballot profile (A_1, A_2, A_3, A_4) with $A_1 = A_2 = \{a, b, c\}$, $A_3 = A_4 = \{a, b, d\}$, and $k = 2$. The committee $\{c, d\}$ provides FPR for this election (we can set $v_{1,c} = v_{2,c} = 1$, $v_{3,d} = v_{4,d} = 1$). However, it does not provide individual representation: the set of all voters forms a 2-cohesive group, but each voter only approves one member of $\{c, d\}$.
- (3) Note that if no group of voters is 1-cohesive, then strong justified representation and individual representation are not binding, so every committee provides both; however, a committee may still fail FPR. Concretely, consider a ballot profile (A_1, A_2) with $A_1 = \{a\}$, $A_2 = \{b\}$, and $k = 1$. For this election, no group of voters is 1-cohesive, so $\{a\}$ provides strong justified representation and individual representation, but not FPR.

□

A key difference between these axioms and PR/FPR is that PR/FPR provide representation guarantees to all voters, whereas the other axioms only provide guarantees to voters belonging to ℓ -cohesive groups (for a suitable value of $\ell \geq 1$).

6.4. Relationship with apportionment axioms

In this section, we explore the relationship between FPR and an axiom for apportionment methods discussed by Balinski and Young [27], called *weak proportionality* [[27], p. 97]. We first need to introduce some terminology.

An *apportionment problem* is given by a vector of non-negative integers $u \in (\mathbb{Z}_+)^p$ and a target number of seats $k \geq 1$; at least one component of u is required to be positive. An apportionment problem captures the ballots cast in a closed-list election: p is understood to be the number of parties that participate in the election, the component u_j of u is the number of votes received by party j , and the goal is to allocate k seats. An *apportionment method* M receives as input an apportionment problem (u, k) and outputs a *seat allocation*, i.e., a vector $S \in (\mathbb{Z}_+)^p$ such that $\sum_{j \in [p]} S_j = k$, where the component S_j of S is the number of seats allocated to party j .

An apportionment method M is said to be *weakly proportional* [[27], p. 97] if for each apportionment problem (u, k) such that there exist a vector $S \in (\mathbb{Z}_+)^p$ satisfying (i) $\sum S_j = k$, and (ii) $u = \lambda S$ for some positive real λ it holds that $M(u, k) = S$. We will say that such S is a *strictly proportional solution* and that the apportionment problem (u, k) *admits a strictly proportional solution*. Balinski and Young [27] proved that, whenever an apportionment problem admits a strictly proportional solution, such a solution is unique.

According to Balinski and Young [27], weak proportionality is one of the “rock-bottom requirements that must be satisfied by any method that is worthy of consideration.”

Brill et al. [29] developed a procedure to transform any approval-based multi-winner voting rule F into an apportionment method M_F . Their procedure executes the following steps: (i) it transforms an apportionment problem (u, k) into an approval-based multi-winner election $(\mathcal{A}, k)$; (ii) it applies the voting rule F to obtain a winning committee W ; and (iii) it derives a seat allocation for (u, k) from W .

In more detail, in step (i), given $u = (u_1, \dots, u_p)$ and k , we create a set of candidates C_j of size k for each party j ; these sets of candidates are pairwise disjoint. The set of candidates of the multi-winner election is then $\cup_{j \in [p]} C_j$. Further, for each $j \in [p]$ we create u_j voters whose approval ballot is C_j . Let $(\mathcal{A}, k)$ be the resulting approval-based multi-winner election; we say that $(\mathcal{A}, k)$ is *induced* by (u, k) . In step (ii), we apply F to $(\mathcal{A}, k)$ to obtain a committee W . In step (iii), for each $j \in [p]$ we allocate $|W \cap C_j|$ seats to party j .

It turns out that if F satisfies FPR, the resulting apportionment method M_F is weakly proportional.

Theorem 9. *If an approval-based multi-winner voting rule F satisfies FPR, then the apportionment method M_F is weakly proportional.*

Proof. Fix an approval-based multi-winner voting rule F that satisfies FPR. Consider an apportionment problem (u, k) such that there exists a vector $S \in (\mathbb{Z}_+)^p$ satisfying (i) $\sum_{j \in [p]} S_j = k$, and (ii) $u = \lambda S$ for some positive real λ . Let (A, k) be the approval-based multi-winner election induced by (u, k) . To show that M_F is weakly proportional, we need to prove that, given (A, k) , the rule F outputs a committee W that satisfies $|W \cap C_j| = S_j$ for all $j \in [p]$.

First, we will argue that a committee W of size k provides FPR for (A, k) if and only if $|W \cap C_j| = S_j$ for all $j \in [p]$. For each $j \in [p]$, let N_j be the set of voters with ballot C_j , and let $n = \sum_{j \in [p]} u_j$.

For the ‘if’ direction, we can set $v_{i,w} = \frac{1}{S_j}$ for each voter $i \in N_j$ and each $w \in W \cap C_j$, and $v_{i,w} = 0$ otherwise; these values satisfy all the conditions in Definition 4 (observe that $u = \lambda S$ and $\sum_{j \in [p]} S_j = k$ implies $\lambda = \frac{n}{k}$).

For the ‘only if’ direction, suppose that W provides FPR, as witnessed by a collection $(v_{i,w})_{i \in [n], w \in W}$. Fix a $j \in [p]$. We have $v_{i,w} = 0$ for each $i \in N_j, w \in W \setminus C_j$, so

$$\sum_{i \in N_j} \sum_{w \in W \cap C_j} v_{i,w} = |N_j| = u_j.$$

On the other hand, by FPR we have

$$\sum_{i \in N_j} v_{i,w} = \frac{n}{k} = \lambda \quad \text{for each } w \in W \cap C_j.$$

It follows that $|W \cap C_j| = u_j / \lambda = S_j$.

Now, by construction, (A, k) admits a committee that provides FPR: e.g., we can take the first S_j candidates from each set C_j , $j \in [p]$. As F satisfies FPR, it must output a committee that provides FPR on (A, k) . But we have argued that every such committee contains exactly S_j candidates from C_j for all $j \in [p]$, which is what we set out to prove. $\square$

In contrast, PR does not imply weak proportionality. Indeed, consider the apportionment problem given by (u, k) , where $u = (17, 34)$ and $k = 9$. Any apportionment method that is weakly proportional must output $S = (3, 6)$ for this apportionment problem. However, since $\frac{n}{k} = \frac{51}{9}$ is not an integer, a voting rule that satisfies PR can output any committee of size k on the approval-based multi-winner election induced by (u, k) .

7. Discussion

Our results highlight a difficulty with the notion of EJRP: this axiom is incompatible with (fractional) perfect representation, which is a very desirable property in parliamentary elections and other settings where fairness is of paramount importance. This perspective is further supported by the importance of weak proportionality in the context of apportionment and the relationship between weak proportionality and FPR established in Section 6.4.

We therefore propose an alternative to the EJRP axiom, Proportional Justified Representation, which is motivated by similar considerations (namely, ensuring that large cohesive groups of voters are allocated several representatives), but does not conflict with PR. PJRP also has further attractive properties: it is satisfied by several well-known multi-winner rules (for some of these rules we have to additionally require that k divides n), some of which are efficiently computable, and, just like EJRP, it provides a justification for using the harmonic weight vector $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ as the default weight vector for PAV.

However, the results of Section 5 can be viewed as an argument in favour of EJRP: every committee that provides EJRP guarantees high levels of average satisfaction to members of large cohesive groups, whereas the guarantee offered by committees that provide PJRP is, in general, much weaker. Thus, one can think of EJRP as a more pragmatic requirement: for every ballot profile a committee that provides EJRP (and, as shown by Aziz et al. [1], such a committee is guaranteed to exist) ensures that members of large cohesive groups are happy on average, at the cost of possibly ignoring other agents. In some applications of multi-winner voting such a trade-off may be acceptable. Consider, for instance, an academic department where members of different research groups pool their funding to run a departmental seminar. Faculty members have preferences over potential speakers, with members of each research group agreeing on a few candidates from their field. Choosing speakers so as to please the members of large research groups may be a good strategy in this case, even if this means that some members of the department will not be interested in any of the talks. Indeed, if very few talks are of interest to members of a large group, this group may prefer to withdraw its contribution to the funding pool and run its own event series.

In contrast, when selection committees that will have to vote on issues, we may prefer voting rules that satisfy PJRP, are efficiently computable, and are close to (F)PR: one such rule is MMS [69], which satisfies PJRP, can be evaluated in polynomial time, and offers a worst-case constant factor approximation to the rule leximax-Phragmén [8] that satisfies PR.

The perfect representation axioms, i.e., PR and FPR, are very different in spirit from JR and its variants: while every election admits a JR committee, there are many elections for which (fractional) perfect representation is not achievable. However, whenever the (F)PR axiom applies, it places very strong constraints on the behaviour of a voting rule, forcing it to choose an (F)PR committee. In contrast, the JR axiom is much easier to satisfy: e.g., for $k = 1$ it suffices to choose a candidate approved by at least one voter. Consequently, while (F)PR implies PJRP at the level of committees (Theorems 3 and 7), this is not the case at the level of voting rules. It is also instructive to compare these axioms from an algorithmic perspective: while JR and its variants can be satisfied by polynomial-time computable voting rules, PR and FPR are fundamentally incompatible with polynomial-time computability (assuming $P \neq NP$). Indeed, it would be interesting to identify scenarios that admit efficient algorithms for computing FPR committees.

CRedit authorship contribution statement

Luis Sánchez-Fernández: Writing – review & editing, Writing – original draft, Investigation; **Edith Elkind:** Writing – review & editing, Writing – original draft, Investigation; **Martin Lackner:** Writing – review & editing, Writing – original draft, Investigation; **Norberto Fernández García:** Writing – original draft, Investigation; **Jesús A. Fisteus:** Writing – review & editing, Writing – original draft, Investigation; **Pablo Basanta Val:** Writing – original draft, Investigation; **Piotr Skowron:** Writing – original draft, Investigation.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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