



Locally conformally flat weakly-Einstein manifolds

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Abstract. It is shown that locally conformally flat weakly-Einstein manifolds are either locally symmetric, and hence a product $N_1^m(c) \times N_2^m(-c)$, or otherwise they are locally homothetic to some specific warped product metrics $\mathcal{I} \times_f N(c)$. As an application we classify weakly-Einstein hypersurfaces in the Euclidean space.

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1. Introduction. Let (M^n, g) be an n -dimensional Riemannian manifold. Let ∇ be the Levi-Civita connection and let R denote the curvature operator taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$. The Ricci tensor $\rho(X, Y) = \text{tr}\{Z \mapsto R(Z, X)Y\}$ and the scalar curvature $\tau = \text{tr}_g \rho$ are widely investigated natural Riemannian invariants. There are other natural symmetric tensor fields defined on (M, g) . For instance $\check{R}_{ij} = R_{i\alpha\beta\gamma} R_j^{\alpha\beta\gamma}$ is a symmetric $(0, 2)$ -tensor field on M which may be considered as the simplest one after the Ricci tensor. However, the condition that $\check{R}$ is a pointwise multiple of the metric tensor,

$$\check{R} = \frac{\|R\|^2}{n} g, \quad (1.1)$$

does not seem to have received much attention in the literature (see, for example, the discussion in [3]). In sharp contrast with Einstein metrics (where the constancy of the scalar curvature τ follows from $\rho = \frac{\tau}{n}g$ if $\dim M \geq 3$), a Schur-like lemma ensuring the constancy of $\|R\|^2$ does not hold in general for

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manifolds satisfying (1.1). It has been proven in [5] that $\|R\|^2$ is constant in dimension $n \geq 5$ provided that the Ricci tensor is a Codazzi tensor.

An n -dimensional Einstein manifold (M, g) is said to be *super-Einstein* if the curvature identity (1.1) is satisfied and $\|R\|^2$ is constant. This curvature condition was already considered by Gray and Willmore in the context of mean-value theorems in Riemannian geometry [12]. The influence of the super-Einstein condition has been investigated in the framework of geodesic spheres by Chen and Vanhecke in [8] and it is related to the existence of constant scalar curvature tangent unit sphere bundles [5]. Furthermore, a compact Einstein metric (or more generally, a metric with parallel Ricci tensor) is critical for the curvature functional $\mathcal{SR} : g \mapsto \int_M \|R\|^2$ restricted to metrics with volume one if and only if (1.1) is satisfied (see [2]).

It was shown in [9] that (1.1) holds true for any four-dimensional Einstein manifold with $\|R\|^2$ not necessarily constant. This led Euh, Park, and Sekigawa to consider the Einstein-like condition (1.1) separately;

Definition 1 [9]. A non-Einstein Riemannian manifold (M, g) is said to be weakly-Einstein if $\tilde{R} = \frac{\|R\|^2}{n}g$.

Our purpose in this paper is to investigate the class of weakly-Einstein Riemannian manifolds aimed to give some new examples and to characterize weakly-Einstein locally conformally flat manifolds (cf. Theorem 2) and weakly-Einstein hypersurfaces in Euclidean spaces (cf. Theorem 12). It is worth to emphasize that dimensions three and four present specific features, and hence, both cases are considered separately in what follows.

2. Locally conformally flat weakly-Einstein manifolds. It is well known that the Ricci tensor completely determines the curvature of a locally conformally flat manifold. In particular, Einstein locally conformally flat manifolds are of constant sectional curvature. The weakly-Einstein condition allows some additional solutions, and our purpose is to prove the following

Theorem 2. *Let (M, g) be a locally conformally flat Riemannian manifold. Then (M, g) is weakly-Einstein if and only if one of the following holds*

- (i) $\dim M = 4$ and (M, g) has vanishing scalar curvature.
- (ii) $\dim M \neq 4$ and either
 - (ii.a) (M, g) is locally homothetic to a warped product $\mathcal{I} \times_f N(c)$, with metric $g = dt^2 + f^2 g_N$, where $\mathcal{I}$ is a real interval and $(N(c), g_N)$ is a manifold of constant sectional curvature $c \in \{0, \pm 1\}$. Furthermore the warping function is given by
 - (ii.a.1) $f(t)^2 = t^2 - 1$ if $c = 1$, and $\mathcal{I} = (1, +\infty)$,
 - (ii.a.2) $f(t)^2 = t$ if $c = 0$, and $\mathcal{I} = (0, +\infty)$,
 - (ii.a.3) $f(t)^2 = 1 - t^2$ if $c = -1$, and $\mathcal{I} = (-1, 1)$ or
 - (ii.b) (M, g) is locally isometric to a product $M = N_1^m(c) \times N_2^m(-c)$, where $m \geq 2$.

Remark 3. Note that, while $\|R\|^2$ is constant in Theorem 2(ii.b), the weakly-Einstein manifolds corresponding to Theorem 2(ii.a) have non-constant $\|R\|^2$.

Remark 4. A Riemannian product $M = N_1^{n_1}(c_1) \times N_2^{n_2}(c_2)$ of two manifolds of constant curvature is weakly Einstein if and only if $c_1^2(n_1 - 1) = c_2^2(n_2 - 1)$. Since the case $c_1 c_2 > 0$ leads to Einstein metrics, the locally conformally flat structure in Assertion (ii.b) of Theorem 2 is a special case of the general situation when $c_1 c_2 < 0$.

The proof of Theorem 2 follows at once from the results in Sections 2.1–2.3. Lemmas 5 and 6 discuss the structure of weakly-Einstein locally conformally flat algebraic curvature tensors. The geometric realizability of these algebraic structures depends on the Schouten tensor, which is a Codazzi tensor. Lemma 9 shows that either the manifold is locally symmetric or it is a warped product. Finally, in Lemma 11 we explicitly determine the form of the corresponding warping functions, thus completing the proof.

2.1. Algebraic structure. We firstly work at a generic point to discuss the algebraic form of the curvature tensor. Let Q be the Ricci operator given by $\rho(X, Y) = g(QX, Y)$, and assume that Q is not a multiple of the identity.

2.1.1. The three-dimensional case.

Lemma 5. *Let (M, g) be a three-dimensional Riemannian manifold. Then it is weakly-Einstein if and only if the Ricci operator is of rank one.*

Proof. Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame of eigenvectors of the Ricci operator, i.e., $Q(e_i) = \lambda_i e_i$, $i = 1, 2, 3$. Since the Weyl tensor vanishes in dimension three, $R_{\alpha ijk} = \frac{1}{2}(\tau - 2(\lambda_\alpha + \lambda_i))\{\delta_{ij}\delta_{\alpha k} - \delta_{\alpha j}\delta_{ik}\}$. Now taking $\alpha, \beta \in \{1, 2, 3\}$, a straightforward calculation shows that $\check{R}(e_\alpha, e_\beta) = 0$ if $\alpha \neq \beta$, and

$$\begin{aligned} \check{R}(e_\alpha, e_\alpha) &= 2 \sum_i R_{\alpha i i \alpha}^2 = 2 \sum_{i \neq \alpha} \left\{ \frac{\tau^2}{4} + (\lambda_\alpha^2 + \lambda_i^2 + 2\lambda_\alpha \lambda_i) - \tau(\lambda_\alpha + \lambda_i) \right\} \\ &= 2 \|\rho\|^2 - \tau^2 + 2\lambda_\alpha(\tau - \lambda_\alpha). \end{aligned}$$

Hence, (M, g) is weakly-Einstein if and only if $\check{R}(e_\alpha, e_\alpha) = \check{R}(e_\beta, e_\beta)$, and thus

$$(\lambda_\alpha - \lambda_\beta)\{\tau - (\lambda_\alpha + \lambda_\beta)\} = 0, \quad \text{for all } \alpha, \beta = 1, 2, 3. \quad (2.1)$$

Now, equation (2.1) holds true if and only if the Ricci operator is a multiple of the identity (in which case (M, g) is Einstein) or it is of rank one. $\square$

2.1.2. The higher dimensional case.

Lemma 6. *An n -dimensional locally conformally flat Riemannian manifold with $n \geq 4$ is weakly-Einstein if and only if one of the following holds*

- (i) $\dim M = 4$ and the scalar curvature vanishes.
- (ii) $\dim M \geq 5$ and the Ricci operator has exactly two-distinct eigenvalues λ and $\mu = -\frac{(n-4)(n-1)+2m}{(n-4)(n-1)+2(n-m)}\lambda$, where m is the multiplicity of the eigenvalue λ .

Remark 7. Assertion (i) in Lemma 6 was already claimed in [10].

Proof. An n -dimensional Riemannian manifold ($n \geq 4$) is locally conformally flat if and only if the Weyl tensor vanishes, in which case

$$\begin{aligned} R(X, Y)Z &= -\frac{\tau}{(n-2)(n-1)}\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + \frac{1}{(n-2)}\{\rho(Y, Z)X - \rho(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\}. \end{aligned} \quad (2.2)$$

Let $\check{Q}$ be the $(1, 1)$ -tensor field defined by $g(\check{Q}(X), Y) = \check{R}(X, Y)$. A straightforward calculation using (2.2) gives

$$\check{Q} = \frac{2}{(n-2)^2} \left\{ (n-4)Q^2 + \frac{2\tau}{(n-1)}Q + \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)}\text{Id} \right\}.$$

Since $\text{tr } \check{Q} = \|R\|^2$ one has

$$\|R\|^2 = \frac{2}{(n-2)^2} \left\{ (n-4)\|\rho\|^2 + \frac{2\tau^2}{(n-1)} + n \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)} \right\}.$$

Recall that (M, g) is weakly-Einstein if and only if $\check{Q} = \frac{\|R\|^2}{n}\text{Id}$. Hence previous expressions show that (M, g) is weakly-Einstein if and only if the Ricci operator satisfies

$$(n-4)Q^2 + \frac{2\tau}{n-1}Q - \left(\frac{(n-4)}{n}\|\rho\|^2 + \frac{2\tau^2}{n(n-1)} \right)\text{Id} = 0. \quad (2.3)$$

If $n = 4$, then (2.3) reduces to $\tau Q - \frac{1}{4}\tau^2\text{Id} = 0$ and hence (M, g) is Einstein or $\tau = 0$. Alternatively, Assertion (i) also follows from the four-dimensional curvature identity $\check{R} - \frac{1}{4}\|R\|^2g = \frac{1}{3}\tau\rho_0 + 2\overset{\circ}{W}[\rho_0]$ (see [13] and [2, Equation (4.72)]), where $\rho_0 = \rho - \frac{1}{4}\tau g$ is the trace-less Ricci tensor, $\overset{\circ}{W}[\rho_0](X, Y) = \sum_i \rho_0(W(e_i, X)Y, e_i)$ and $\{e_i\}$ is an orthonormal basis.

If $n \geq 5$, then the Ricci curvatures necessarily satisfy (2.3), which shows that the number of different Ricci curvatures is at most two. On the other hand, since (M, g) is not Einstein the Ricci operator has necessarily two-distinct eigenvalues λ, μ in an open-dense set. Then it follows from Vieta's formulae that

$$\lambda + \mu = -\frac{2\tau}{(n-4)(n-1)}.$$

Let m denote the multiplicity of the Ricci curvature λ . Then $\tau = m\lambda + (n-m)\mu$ and thus

$$\mu = -\frac{2m + (n-4)(n-1)}{2(n-m) + (n-4)(n-1)}\lambda,$$

which proves Assertion (ii). $\square$

2.2. Underlying structure. Let $S = \frac{1}{n-2}(\rho - \frac{\tau}{2(n-1)}g)$ be the Schouten tensor of (M, g) . A three-dimensional Riemannian manifold is locally conformally flat if and only if the Schouten tensor is Codazzi, i.e., its covariant derivative is totally symmetric. Furthermore since the Cotton tensor $C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik}$ is given by the divergence of the Weyl tensor, one has that the Schouten tensor of any locally conformally flat manifold is a Codazzi tensor. Moreover, it follows from Lemmas 5 and 6 that the Schouten tensor of any locally conformally flat weakly-Einstein manifold has exactly two-distinct eigenfunctions provided that $\dim M \neq 4$.

We recall some facts about Codazzi tensors.

Lemma 8 [4, 14]. *Let T be a Codazzi tensor on a Riemannian manifold (M^n, g) , $n \geq 3$. Let λ be an eigenfunction of T with associated eigenspace V_λ . If $\dim V_\lambda \geq 2$, then $\nabla \lambda$ is orthogonal to V_λ . Moreover, if T has exactly two-distinct eigenfunctions λ and μ with $\dim V_\lambda \leq \dim V_\mu$, then*

- (i) M is locally a product if $\dim V_\lambda \geq 2$.
- (ii) M is locally a warped product with one-dimensional base and non-trivial warping function if and only if
 - (ii.a) $\dim V_\lambda = 1$,
 - (ii.b) the eigenfunction μ is not constant and $\nabla \lambda$ is orthogonal to V_μ .

Now, we have

Lemma 9. *Let (M^n, g) be a weakly-Einstein locally conformally flat Riemannian manifold with $n \neq 4$. Then*

- (i) *If the Ricci curvatures λ and μ have multiplicity greater than one, then (M, g) is locally isometric to a product $M = N_1^m(c) \times N_2^m(-c)$.*
- (ii) *If the Ricci curvature λ has multiplicity one, then (M, g) is locally isometric to a warped product of the form $\mathbb{R} \times_f F$, where (F, g_F) is of constant curvature.*

Proof. Assume $\dim M \geq 5$. It follows from the proof of Lemma 6 that the scalar curvature of any weakly-Einstein locally conformally flat manifold in dimension ≥ 5 is given by

$$\tau = -\frac{(n-4)(n-1)(n-2m)}{(n-4)(n-1) + 2(n-m)}\lambda.$$

Now, a straightforward calculation from Lemma 6 shows that the Schouten tensor has exactly two-distinct eigenvalues

$$\bar{\lambda} = \frac{2m-3n+4}{4m-2n^2+6n-8}\lambda, \quad \bar{\mu} = \frac{2m+n-4}{4m-2n^2+6n-8}\lambda = \frac{2m+n-4}{2m-3n+4}\bar{\lambda},$$

where m is the multiplicity of the eigenvalue $\bar{\lambda}$.

Assume $\dim V_{\bar{\lambda}} \leq \dim V_{\bar{\mu}}$. If $\dim V_{\bar{\lambda}} \geq 2$, then (M, g) is locally a product by Lemma 8. Local conformal flatness shows that $M = N_1(c) \times N_2(-c)$ where $N_i(\kappa)$ is a manifold of constant sectional curvature κ (see, for example, [18, Theorem 4]). Let $m = \dim N_1$. Remark 4 shows that $n-m-1 = m-1$ and thus $n = 2m$, which proves Assertion (i). Also, if any of the Schouten eigenfunctions $\bar{\lambda}$, $\bar{\mu}$ is constant so are the Ricci curvatures λ and μ . Since (M, g) is locally

conformally flat with constant Ricci curvatures, it is curvature homogeneous and hence locally symmetric by [17]. Then (M, g) splits as a product whose factors correspond to the different Ricci curvatures and (M, g) is locally a product as in Assertion (i).

Assume now $\dim V_{\bar{\lambda}} = 1$, $\dim V_{\bar{\mu}} = n - 1 \geq 2$. Then, by Lemma 8, $\nabla \bar{\mu}$ is orthogonal to $V_{\bar{\mu}}$, and thus (since $\bar{\lambda}$ and $\bar{\mu}$ only differ in a multiplicative constant), one has that $\nabla \bar{\lambda}$ is also orthogonal to $V_{\bar{\mu}}$. Further since the eigenvalue $\bar{\mu}$ is not constant, Lemma 8(ii) shows that (M, g) is a warped product with one-dimensional basis. Moreover, the fiber of the warped product is of constant sectional curvature due to local conformal flatness (see [6]) and Assertion (ii) follows.

Assume now $\dim M = 3$. By Lemma 5 the Ricci operator is of rank one, i.e., $Q = \text{diag}[0, 0, \mu]$ for some function μ . Hence the Schouten tensor $S = Q - \frac{\tau}{4} \text{Id}$ has eigenvalues $\bar{\lambda} = -\frac{1}{4}\mu$ and $\bar{\mu} = \frac{3}{4}\mu$, the later with multiplicity one. Let $V_{\bar{\lambda}}$ and $V_{\bar{\mu}}$ be the corresponding eigenspaces. Since $\dim V_{\bar{\lambda}} = 2$ and S is a Codazzi tensor, one has that $\nabla \bar{\lambda} = -\frac{1}{4}\nabla \mu$ is orthogonal to $V_{\bar{\lambda}}$ by Lemma 8. Moreover, the Ricci curvature μ is not constant unless it vanishes. Indeed, if μ is constant, then (M, g) is 0-curvature homogeneous and thus locally symmetric by [17]. Therefore the Ricci tensor is parallel and (M, g) splits as a product, which is a contradiction with the fact that Q is of rank one. Since the gradients of the two eigenvalues $-\frac{1}{4}\mu$ and $\frac{3}{4}\mu$ are collinear, one has that $\nabla \bar{\mu} = \frac{3}{4}\nabla \mu$ is also orthogonal to $V_{\bar{\lambda}}$. Then by Lemma 8(ii) one has that (M, g) is locally a warped product of the form $\mathbb{R} \times_f F$ for some surface (F, g_F) . Now, the warped product metric is in the conformal class of a product metric since $dt \otimes dt + f(t)^2 g_F = f(t)^2 \{f(t)^{-2} dt \otimes dt + g_F\}$. Hence the product metric $f(t)^{-2} dt \otimes dt + g_F$ is locally conformally flat and thus the corresponding Schouten tensor is Codazzi. Since $\dim F = 2$, its Ricci operator satisfies $Q^F = \frac{1}{2}\tau^F \text{Id}$, and thus the Codazzi tensor of the product metric $(M, f(t)^{-2} dt \otimes dt + g_F)$ is given by $S = \tau^F \text{diag}[-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$. Set $X = \partial_t$ and take Y to be a vector field on M tangent to F . Now it follows from the condition $(\nabla_X S)Y = (\nabla_Y S)X$ that the scalar curvature τ^F is constant. Hence, since $\dim F = 2$, the fiber (F, g_F) is of constant sectional curvature. This completes the proof of Assertion (ii). $\square$

2.3. Weakly-Einstein warped products. We refer to [16] for basic facts and the expressions of the curvature tensor of warped product metrics. Let (B, g_B) and (F, g_F) be Riemannian manifolds. Let $M = B \times F$, and let $g = g_B + f^2 g_F$ be the warped product metric on M for any positive or negative function $f \in C^\infty(B)$. Let $H_f = \nabla df$ be the Hessian tensor and h_f , given by $g(h_f X, Y) = H_f(X, Y)$, the corresponding Hessian operator. Then one has

Lemma 10. *Let $B \times_f F$ be a warped product manifold with $\dim F = d$. Then*

- (i) $\check{R}(X, Y) = \check{R}^B(X, Y) + \frac{2d}{f^2} \|h_f\|^2 g(X, Y)$,
- (ii) $\check{R}(X, V) = 0$,
- (iii) $\check{R}(U, V) = \frac{1}{f^2} \check{R}^F(U, V) - 4 \frac{\|\nabla f\|^2}{f^2} \rho^F(U, V) + 2 \{ \frac{\|h_f\|^2}{f^2} + (d-1) \frac{\|\nabla f\|^4}{f^4} \} g(U, V)$,

where X, Y, Z (resp., U, V, W) are lifts to M of vector fields on B (resp., vector fields on F).

Proof. The proof follows by standard calculations using the expressions of the curvature tensor of warped product metrics in [16]. $\square$

We are mainly interested in the locally conformally flat case, thus assuming the fiber (F, g_F) to be of constant sectional curvature and the base manifold a real interval $\mathcal{I} \subset \mathbb{R}$.

Lemma 11. *A warped product $\mathcal{I} \times_f N$ with fiber $N(c)$ of constant sectional curvature c is weakly Einstein if and only if it is homothetic to one of the following*

- (i) $f(t)^2 = t^2 - 1$ if $c = 1$, and $\mathcal{I} = (1, +\infty)$,
- (ii) $f(t)^2 = t$ if $c = 0$, and $\mathcal{I} = (0, +\infty)$,
- (iii) $f(t)^2 = 1 - t^2$ if $c = -1$, and $\mathcal{I} = (-1, 1)$.

Proof. Let (N, g_N) be an $(n-1)$ -dimensional manifold of constant sectional curvature c . Then $\rho^N = c(n-2)g_N$ and $\check{R}^N = 2c^2(n-2)g_N$. A direct application of Lemma 10 shows that $M = \mathbb{R} \times_f N(c)$ is weakly-Einstein if and only if

$$2(n-1) \frac{1}{f^2} (f'')^2 = \frac{2}{f^2} \left\{ \frac{c^2}{f^2} (n-2) - 2c(n-2) \frac{(f')^2}{f^2} + (f'')^2 + (n-2) \frac{(f')^4}{f^2} \right\},$$

from where one gets $0 = (f')^4 - f^2(f'')^2 - 2c(f')^2 + c^2$. Now, write

$$\begin{aligned} & f'(t)^4 - f(t)^2 f''(t)^2 - 2c f'(t)^2 + c^2 \\ &= (f'(t)^2 - f(t) f''(t) - c) (f'(t)^2 + f(t) f''(t) - c). \end{aligned}$$

to decouple the above equation into the two equations

$$f'(t)^2 - f(t) f''(t) - c = 0, \quad \text{and} \quad f'(t)^2 + f(t) f''(t) - c = 0.$$

A standard calculation shows that the solutions of $f'(t)^2 - f(t) f''(t) - c = 0$ are given by $f(t) = \pm \left(\frac{1}{2} e^{-e^a(b+t)-2a} - \frac{1}{2} c e^{e^a(b+t)} \right)$ if $c \neq 0$ and $f(t) = a e^{bt}$ if $c = 0$, for arbitrary constants $a, b \in \mathbb{R}$. Furthermore $\mathcal{I} \times_f N$ becomes an Einstein manifold in both situations and we discard this case. The solutions of the second equation above, $f'(t)^2 + f(t) f''(t) - c = 0$, are of the form

$$f(t) = \pm \frac{\sqrt{c^2(b+t)^2 - e^{2a}}}{\sqrt{c}}, \quad \text{for } c \neq 0, \quad \text{and} \quad f(t) = a\sqrt{2t-b}, \quad \text{for } c = 0,$$

where $a, b \in \mathbb{R}$. Finally, a standard calculation shows that the Ricci operator of $\mathcal{I} \times_f N$ satisfies $Q = -\frac{1}{2} \left(\frac{e^{2a} - c^2(b+t)^2}{2ce^a} \right)^2 \text{diag}[-2, 1, \dots, 1]$, and Assertions (ii.a.1) and (ii.a.3) in Theorem 2 follow after rescaling. $\square$

3. Weakly-Einstein hypersurfaces in Euclidean spaces. Einstein hypersurfaces in Euclidean spaces are either totally umbilical, the rank of the shape operator is ≤ 1 at each point, or otherwise they are open parts in the product of two suitable spheres (see [11]). The weakly-Einstein condition is much more rigid as the following shows.

Theorem 12. *A hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ is weakly-Einstein if and only if it is homothetic to an open part of a warped product $\mathcal{I} \times_f N(c)$, where $N(c)$ is a manifold of constant curvature c and the warping function is given by Theorem 2(ii.a).*

Lemma 13. *Let $M \hookrightarrow \mathbb{R}^{n+1}$ be a hypersurface. Then M is weakly Einstein if and only if the shape operator S has exactly two-distinct principal curvatures $\pm\lambda$, i.e., $S = \text{diag}[\lambda, \overset{m}{\cdot}, \lambda, -\lambda, \overset{n-m}{\cdot}, -\lambda]$.*

Proof. Let M be a hypersurface in $\mathbb{R}^{n+1}$ and let $\mathcal{S} = -\bar{\nabla}N$ denote the shape operator, where N is a unit normal vector field. Let $\lambda_1, \dots, \lambda_n$ be the principal curvatures and let $\{e_1, \dots, e_n\}$ denote the corresponding principal directions. Using the Gauss equation

$$R(X, Y, Z, T) = g(SY, Z)g(SX, T) - g(SX, Z)g(SY, T),$$

a standard calculation gives

$$\check{R}(e_i, e_i) = 2\lambda_\alpha^2(\|\mathcal{S}\|^2 - \lambda_\alpha^2), \quad \text{and} \quad \check{R}(e_i, e_j) = 0, \quad \text{for all } i \neq j.$$

Hence (M, g) is a weakly-Einstein hypersurface if and only if

$$0 = \check{R}(e_i, e_i) - \check{R}(e_j, e_j) = 2(\lambda_i^2 - \lambda_j^2)\{\|\mathcal{S}\|^2 - (\lambda_i^2 + \lambda_j^2)\},$$

for all $i, j = 1, \dots, n$. Let m_k denote the multiplicity of the principal curvature λ_k . Then previous equation becomes

$$0 = (\lambda_i - \lambda_j)(\lambda_i + \lambda_j) \left\{ (m_i - 1)\lambda_i^2 + (m_j - 1)\lambda_j^2 + \sum_{i \neq k \neq j} m_k \lambda_k^2 \right\}.$$

Hence the shape operator is either of rank-one or it must have exactly two-distinct principal curvatures $\pm\lambda$. If $\mathcal{S}$ is of rank-one, then M is flat and thus Einstein, which completes the proof. $\square$

Proof of Theorem 12. Since $\mathcal{S}$ is a Codazzi tensor, Lemma 8 shows that the principal curvatures are constant provided that $m \geq 2$ and $n - m \geq 2$. Hence M is an isoparametric hypersurface in an Euclidean space, and thus M is an open part of a flat hyperplane, a metric sphere, or a spherical cylinder $\mathbb{S}^m \times \mathbb{R}^{n-m}$ (see, for example, [7]). Consequently, $\mathcal{S}$ is either a multiple of the identity, in which case M is Einstein, or otherwise $\mathcal{S} = \text{diag}[\lambda, \overset{m}{\cdot}, \lambda, 0, \overset{n-m}{\cdot}, 0]$, which does not correspond to the possibilities in Lemma 13. Therefore one of the principal curvatures must have multiplicity one. In this case, results of [15] show that M is locally conformally flat.

Lemma 9 shows that M is locally a warped product $\mathbb{R} \times_f N(c)$ with fiber (N, g_N) of constant curvature c if $\dim M \neq 4$. If $n = 4$, then the scalar curvature $\tau = 0$ by Lemma 6(i). Hence the Schouten tensor $\mathcal{S}$ reduces to the Ricci tensor, which is necessarily Codazzi. Computing the Ricci operator from the Gauss equation, one has $Q = (\text{tr } \mathcal{S})\mathcal{S} - \mathcal{S}^2$. Therefore the Ricci operator takes the form $Q = \text{diag}[-3\lambda^2, \lambda^2, \lambda^2, \lambda^2]$ which shows that there are exactly two-distinct Ricci curvatures. Since the principal curvatures are not constant, Lemma 8 shows that M has the local structure of a warped product as in Assertion (ii.a) of Theorem 2.

Finally, let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a local orthonormal frame on $M = \mathbb{R} \times_f N(c)$ so that $\mathbf{e}_1$ is tangent to the base of the warped product and $\mathbf{e}_2, \dots, \mathbf{e}_n$ are tangent to the fiber N . Then, an explicit calculation of the sectional curvature of $\mathbb{R} \times_f N(c)$ shows that

$$R(\mathbf{e}_1, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_1) = -\frac{f''(t)}{f(t)}, \text{ and } R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_i) = \frac{c - f'(t)^2}{f(t)^2} \text{ for } i, j = 2, \dots, n.$$

On the other hand, since the shape operator satisfies $\mathcal{S}\mathbf{e}_1 = \lambda\mathbf{e}_1$, $\mathcal{S}\mathbf{e}_i = -\lambda\mathbf{e}_i$, the Gauss equation gives $R(\mathbf{e}_1, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_1) = -\lambda^2$ and $R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_i) = \lambda^2$ for all $i, j = 2, \dots, n$. Hence the warping function $f(t)$ satisfies $f''(t)f(t) + f'(t)^2 - c = 0$, the equation determining the weakly-Einstein spaces in Lemma 11. $\square$

Remark 14. Let $M \hookrightarrow \mathbb{M}(\kappa)^{n+1}$ be a hypersurface in a space of constant curvature $\kappa \neq 0$. It follows from the Gauss equation that M is weakly-Einstein if and only if the shape operator satisfies

$$\begin{aligned} \mathcal{S}^4 - (\|\mathcal{S}\|^2 - 2\kappa)\mathcal{S}^2 - (2n\kappa H)\mathcal{S} \\ - \frac{1}{n}\{\|\mathcal{S}^2\|^2 - (\|\mathcal{S}\|^2 - 2\kappa)\|\mathcal{S}\|^2 - 2\kappa(nH)^2\}\text{Id} = 0, \end{aligned} \quad (3.1)$$

where $H = \frac{1}{n} \text{tr } \mathcal{S}$ is the mean curvature. Hence there are at most four-distinct principal curvatures.

Assume that the shape operator has exactly two-distinct principal curvatures λ and μ with λ of multiplicity one (i.e., $\mathcal{S} = \text{diag}[\lambda, \mu, \mu, \dots, \mu]$). Let $\mathbf{e}_1$ (resp., $\mathbf{e}_i$, $i = 2, \dots, n$) be a principal direction for λ (resp., μ). Then the weakly-Einstein condition gives

$$\begin{aligned} 0 &= \check{R}(\mathbf{e}_1, \mathbf{e}_1) - \check{R}(\mathbf{e}_i, \mathbf{e}_i) \\ &= 2(\lambda - \mu)\{2n\kappa H + (\lambda + \mu)[\|\mathcal{S}\|^2 - 2\kappa - (\lambda^2 + \mu^2)]\}. \end{aligned}$$

Moreover, since $\|\mathcal{S}\|^2 = \lambda^2 + (n-1)\mu^2$ and $H = \frac{1}{n}(\lambda + (n-1)\mu)$, one has that the principal curvatures are related by

$$\mu^2 + \lambda\mu + 2\kappa = 0. \quad (3.2)$$

Results in [15] show that M is locally conformally flat and thus locally a warped product as in Theorem 2(ii.a) if $\dim M \neq 4$. Moreover, if $\dim M = 4$, since the scalar curvature vanishes (cf. Theorem 2(i)), the Ricci tensor is Codazzi. Furthermore, the Gauss equation together with the relation (3.2) show that the Ricci operator takes the form $Q = \text{diag}[-3(\kappa + \mu^2), \kappa + \mu^2, \kappa + \mu^2, \kappa + \mu^2]$, and proceeding as in the proof of Theorem 12, one has that M is locally a warped product $\mathcal{I} \times_f N(c)$ as in Theorem 2 (ii.a).

Contrary to the situation in Theorem 12, the Gauss equation now gives

$$\begin{aligned} R(\mathbf{e}_1, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_1) &= -\frac{f''(t)}{f(t)} = \kappa - \lambda\mu, \quad \text{and} \\ R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_i) &= \frac{c - f'(t)^2}{f(t)^2} = \kappa - \mu^2 \end{aligned}$$

for all $i \neq j$, $i, j = 2, \dots, n$. Adding the two expressions above one has

$$\mu^2 + \lambda\mu - 2\kappa = \frac{1}{f(t)^2} (f''(t)f(t) + f'(t)^2 - c).$$

Hence, Lemma 11 and equation (3.2) show that $\kappa = 0$, which is a contradiction. This shows that *there are no weakly-Einstein hypersurfaces in $\mathbb{M}(\kappa)^{n+1}$, $\kappa \neq 0$, with exactly two-distinct principal curvatures, one with multiplicity one.*

In contrast with the previous situation, if both principal curvatures have multiplicity greater than one, then Lemma 8 shows that M is a isoparametric hypersurface, since the shape operator is a Codazzi tensor. Isoparametric hypersurfaces in the sphere with exactly two-distinct principal curvatures are open parts of spherical cylinders $\mathbb{S}^m(\kappa) \times \mathbb{R}^{n-m}$ (see [7]), and none of them is weakly-Einstein by Remark 4. On the contrary the products $\mathbb{S}^m(\kappa_1) \times \mathbb{H}^{n-m}(\kappa_2)$ with $\kappa_1 = \sinh^{-2} \theta$, $\kappa_2 = -\cosh^{-2} \theta$ are isoparametric hypersurfaces in $\mathbb{H}^n$ (see [7]). It now follows from Remark 4 that $\mathbb{S}^m(\kappa_1) \times \mathbb{H}^{n-m}(\kappa_2)$ is weakly-Einstein if and only if $\tanh^4 \theta = \frac{n-m-1}{m-1}$. Hence, *for any $\frac{n}{2} < m < n-1$, the product manifolds $\mathbb{S}^m(\sinh^{-2} \theta) \times \mathbb{H}^{n-m}(-\cosh^{-2} \theta)$ with $\theta = \operatorname{arctanh} \frac{n-m-1}{m-1}$, are weakly-Einstein hypersurfaces in the hyperbolic space with exactly two-distinct principal curvatures.*

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