

FOUR-DIMENSIONAL HOMOGENEOUS MANIFOLDS SATISFYING SOME EINSTEIN-LIKE CONDITIONS

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ABSTRACT. We classify four dimensional homogeneous manifolds in the setting of $\check{\rho}$ and $R[\rho]$ conditions. We found that there is just a non trivial example in $\mathbb{R} \ltimes H^3$ and that there are not other case in the other groups of the same kind.

1. INTRODUCTION

Let (M^n, g) be an n -dimensional Riemannian manifold. Let ∇ be the Levi-Civita connection and let R be the curvature operator chosen with the sign $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$. With this convention, the Ricci tensor is given by $\rho(X, Y) = \text{tr}\{Z \mapsto R(Z, X)Y\}$ and the scalar curvature $\tau = \text{tr}_g \rho$.

Let M be a compact surface. The Gauss-Bonnet Theorem shows that the functional $g \mapsto \int_M \tau \text{vol}_g$ is constant on the space of metrics and thus the Ricci tensor of any two-dimensional Riemannian manifold (M, g) satisfies the universal identity $\rho = \frac{1}{2}\tau g$. Generalizations of the Gauss-Bonnet Theorem to higher dimensions provide new universal identities. Indeed, the functional defined by the Gauss-Bonnet integrand $g \mapsto \int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\} \text{vol}_g$ is constant in dimension four. Hence any compact four-dimensional Riemannian manifold is critical and Berger [3] showed that the curvature identity $\check{R} - 2\check{\rho} - 2R[\rho] + \tau\rho = \frac{1}{4}(\|R\|^2 - 4\|\rho\|^2 + \tau^2)g$ holds. (See [8] for an extension to the noncompact case and [13] for a generalization to higher dimensions).

Previous identity can be equivalently written as

$$(1.1) \quad \left(\check{R} - \frac{\|R\|^2}{4}g \right) + \tau \left(\rho - \frac{\tau}{4}g \right) - 2 \left(\check{\rho} - \frac{\|\rho\|^2}{4}g \right) - 2 \left(R[\rho] - \frac{\|\rho\|^2}{4}g \right) = 0,$$

where $\check{R}$, $\check{\rho}$ and $R[\rho]$ are the symmetric $(0, 2)$ -tensor fields given by $\check{R}_{ij} = R_{iabc}R_j^{abc}$, $\check{\rho}_{ij} = \rho_{ia}\rho_j^a$ and $R[\rho]_{ij} = R_{iabj}\rho^{ab}$. It now follows that if (M, g) is Einstein, then all terms in (1.1) vanish. Hence it is a natural question to investigate the tensor fields $\check{R}$, $\check{\rho}$ and $R[\rho]$ separately, in particular whether they are multiples of the metric tensor. (M, g) is said to be *weakly-Einstein* if $\check{R} = \frac{1}{4}\|R\|^2g$. (M, g) is said to be *$\check{\rho}$ -Einstein* if $\check{\rho} = \frac{1}{4}\|\rho\|^2g$ and it is said to be *$R[\rho]$ -Einstein* if $R[\rho] = \frac{1}{4}\|\rho\|^2g$.

The weakly-Einstein, $\check{\rho}$ -Einstein and $R[\rho]$ -Einstein conditions appear naturally in the study of critical metrics for quadratic curvature functionals. For instance, a metric with harmonic Ricci tensor ($\Delta\rho = 0$) and constant scalar curvature is

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critical for the functional given by the L^2 -norm of the Ricci tensor if and only if it is $R[\rho]$ -Einstein. Under the same conditions, critical points of the L^2 -norm of the curvature tensor are characterized by the condition $\check{R} + 2R[\rho] - 2\check{\rho} = \frac{1}{4}\|R\|^2 g$ (see, for example [4]).

It is clear from (1.1) that any four-dimensional Einstein metric is weakly-Einstein, $\check{\rho}$ -Einstein and $R[\rho]$ -Einstein. There are however strict examples. For instance, weakly-Einstein locally conformally flat examples with non-constant scalar curvature have been constructed in [11]. The simplest example with constant scalar curvature is the product manifold $M_1(c) \times M_2(-c)$ of two surfaces with constant opposite curvature. We show that it is the only one in the symmetric realm (Lemma 2).

Homogeneous Einstein metrics in dimension four were classified by Jensen [12], showing that they are either real space forms, complex space forms, or a product of two surfaces with the same constant Gauss curvature. The aim of this paper is to extend Jensen's classification to the $\check{\rho}$ -Einstein and $R[\rho]$ -Einstein cases (we refer to [1] for a classification of homogeneous weakly-Einstein four-manifolds). Our main result is the following.

Theorem 1. *Let (M, g) be a four-dimensional homogeneous manifold. Then*

- (1) *(M, g) is weakly-Einstein and non-symmetric if and only if it is homothetic to the Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ with left-invariant metric determined by the Lie algebra*

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3,$$

- (2) *(M, g) is $\check{\rho}$ -Einstein and non-symmetric if and only if it is homothetic to one of the following:*

- (2.a) *The Lie group $SU(2) \times \mathbb{R}$ with left-invariant metric determined by the Lie algebra*

$$\begin{aligned} [e_1, e_2] &= (4 \pm 2\sqrt{2})e_3, & [e_2, e_3] &= (3 \pm 2\sqrt{2})e_1, & [e_3, e_1] &= e_2, \\ [e_4, e_1] &= -e_2, & [e_4, e_2] &= (3 \pm 2\sqrt{2})e_1, \end{aligned}$$

- (2.b) *The Lie group $\mathbb{R} \ltimes H^3$ with left-invariant metric determined by the Lie algebra*

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \frac{1}{2}e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2,$$

- (2.c) *The Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ with left-invariant metric determined by the Lie algebra*

$$\begin{aligned} [e_4, e_1] &= e_1 - \frac{\alpha(1 + \alpha + \alpha^2)}{(\alpha + 1)^2(\alpha - 1)}e_2 + \frac{\alpha(1 + \alpha + \alpha^2)}{2\alpha + 1}e_3, \\ [e_4, e_2] &= \frac{\alpha(1 + \alpha + \alpha^2)}{(\alpha + 1)^2(\alpha - 1)}e_1 + \alpha e_2 + \frac{(1 + \alpha + \alpha^2)^2}{\alpha(\alpha + 2)}e_3, \\ [e_4, e_3] &= -\frac{\alpha(1 + \alpha + \alpha^2)}{2\alpha + 1}e_1 - \frac{(1 + \alpha + \alpha^2)^2}{\alpha(\alpha + 2)}e_2 - \frac{\alpha}{\alpha + 1}e_3, \end{aligned}$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis and $\alpha \in (-1, 1)$, $\alpha \neq -\frac{1}{2}$, $\alpha \neq 0$.

- (3) *(M, g) is $R[\rho]$ -Einstein if and only if it is symmetric.*

The paper is organized as follows. The symmetric case is analyzed in Section 2, thus reducing the problem to the consideration of left-invariant metrics on Lie groups. The $\check{\rho}$ -Einstein condition is examined in Section 3 while the $R[\rho]$ -Einstein

condition is treated in Section 4. Finally the proof of Theorem 1 is given in Section 5.

2. PRELIMINARIES

Bérard-Bergery showed in [2] that any complete and simply connected four dimensional homogeneous space is either symmetric or a Lie group with a left-invariant metric.

Lemma 2. *Let (M, g) be a non-Einstein locally symmetric four-dimensional manifold. Then the following are equivalent*

- (i) (M, g) is weakly-Einstein.
- (ii) (M, g) is $\check{\rho}$ -Einstein.
- (iii) (M, g) is $R[\rho]$ -Einstein.

Moreover, any of the above hold if and only if (M, g) is homothetic to the product $M = \mathbb{S}^2 \times \mathbb{H}^2$.

Proof. Let (M, g) be a symmetric space. Since the Ricci tensor is parallel, the Ricci curvatures are constant and the corresponding eigenspaces are parallel. Let $\{\kappa_1, \dots, \kappa_4\}$ be the Ricci curvatures. If (M, g) is non-Einstein, then there are at least two-distinct Ricci curvatures. Let κ be a Ricci curvature appearing with multiplicity one. Then the associated eigenspace splits off a one-dimensional manifold so that (M, g) is locally isometric to $\mathbb{R} \times N$ for some three-dimensional symmetric manifold N . Now such a product $\mathbb{R} \times N$ (with (N, g_N) symmetric) is weakly-Einstein, $\check{\rho}$ -Einstein or $R[\rho]$ -Einstein if and only if it is flat. Hence the only possibility is that the Ricci operator has exactly two-distinct Ricci curvatures with multiplicity two. Moreover, in such a case a standard calculation shows that the Ricci curvatures κ_1 and κ_2 satisfy $\kappa_1 = -\kappa_2$ and the result follows. $\square$

Hence we focus on left-invariant metrics in what remains of this paper. In order to classify $\check{\rho}$ -Einstein and $R[\rho]$ -Einstein metrics, one needs to solve some systems of polynomial equation on the structure constants. Whenever a system of polynomial equations is simple, finding all common roots may be an easy task, but if the number of polynomials and their degree increases, it usually becomes an unmanageable task. Given $\mathcal{S}$ a system of polynomials $\mathfrak{P}_i \in \mathbb{R}[x_1, \dots, x_n]$, a n -tuple $\vec{x}$ is a solution for $\mathcal{S}$ if and only if $\mathfrak{P}_i(\vec{x}) = 0$, for all i . It is a fundamental observation that $\vec{x}$ is a solution for the system if and only if it is a solution of the ideal $\mathcal{I} = \langle \mathfrak{P}_i \rangle$ generated by the polynomials of the system.

Note that if two sets of polynomials generate the same ideal, the corresponding zero sets must be identical. Hence one may try to simplify the problem looking for “better” polynomials in the ideal, since the zero sets are the same. Gröbner basis provide an useful tool in solving systems of polynomial equations by following the above strategy.

Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha \in \mathbb{Z}_{\geq 0}^n$ be a monomial in $\mathbb{R}[x_1, \dots, x_n]$. Establishing an ordering on $\mathbb{Z}_{\geq 0}^n$ will induce an ordering on the monomials. We are specially interested in the lexicographical ordering: We say that $\alpha >_{lex} \beta$ if in the vector $\alpha - \beta \in \mathbb{Z}^n$, the leftmost non-zero entry is positive. Ordering the monomials of any polynomial $\mathfrak{P} = \sum_{\alpha} a_{\alpha} x^{\alpha}$ in $\mathbb{R}[x_1, \dots, x_n]$ as before, the *multidegree* of $\mathfrak{P}$ is the maximum $\alpha \in \mathbb{Z}_{\geq 0}^n$ so that $a_{\alpha} \neq 0$, where the maximum is taken with respect to the lexicographical monomial ordering. The corresponding monomial is called the *leading term* $LT(\mathfrak{P}) = a_{\alpha} x^{\alpha}$.

Let $\mathcal{I} \subset \mathbb{R}[x_1, \dots, x_n]$ be a non-zero ideal. Let $LT(\mathcal{I})$ be the set of leading terms of all elements of $\mathcal{I}$ and let $\langle LT(\mathcal{I}) \rangle$ be the ideal generated by the elements of $LT(\mathcal{I})$. It is important to emphasize that if $\mathcal{I} = \langle \mathfrak{P}_1, \dots, \mathfrak{P}_k \rangle$, then $\langle LT(\mathcal{I}) \rangle$ may be strictly larger than the ideal $\langle LT(\mathfrak{P}_1), \dots, LT(\mathfrak{P}_k) \rangle$. A finite subset $\mathcal{G} = \{\mathbf{g}_1, \dots, \mathbf{g}_\nu\}$ of an ideal $\mathcal{I}$ is said to be a *Gröbner basis* with respect to the lexicographical monomial order if $\langle LT(\mathbf{g}_1), \dots, LT(\mathbf{g}_\nu) \rangle = \langle LT(\mathcal{I}) \rangle$. The Hilbert Basis Theorem guarantees that any non-zero ideal $\mathcal{I} \subset \mathbb{R}[x_1, \dots, x_n]$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal $\mathcal{I}$ is a basis of $\mathcal{I}$.

Buchberger's algorithm (among others) provides a constructive algorithm to find one such basis. Roughly speaking, it proceeds as follows

- (1) Given an ideal $\mathcal{I} \subset \mathbb{R}[x_1, \dots, x_n]$. Choose $\mathbf{g}_i, \mathbf{g}_j \in \mathcal{I}$ with multidegrees α_i and α_j respectively.
- (2) Calculate $S(\mathbf{g}_i, \mathbf{g}_j) = \frac{x^\gamma}{LT(\mathbf{g}_i)}\mathbf{g}_i - \frac{x^\gamma}{LT(\mathbf{g}_j)}\mathbf{g}_j$, where $\gamma = \max(\alpha_i, \alpha_j)$.
- (3) Compute the rest of the division of $S(\mathbf{g}_i, \mathbf{g}_j)$ by all the polynomials in $\mathcal{I}$. If the rest is nonzero, add $S(\mathbf{g}_i, \mathbf{g}_j)$ to $\mathcal{I}$, and repeat until all rests are zero.

We refer to [10] for more information on Gröbner basis.

3. LEFT-INVARIANT $\check{\rho}$ -EINSTEIN METRICS

Since any non-symmetric homogeneous four-dimensional manifold is a Lie group with a left-invariant metric, in this section we analyze the existence of $\check{\rho}$ -Einstein metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$, and the solvable cases corresponding to the semi-direct products $\mathbb{R}e_4 \ltimes H^3$, $\mathbb{R}e_4 \ltimes E(1, 1)$, $\mathbb{R}e_4 \ltimes E(2)$, and $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

3.1. Left-invariant $\check{\rho}$ -Einstein metrics in $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$.

Lemma 3. *$SU(2) \times \mathbb{R}$ admits a non-Einstein $\check{\rho}$ -Einstein left invariant metric if and only if it is homothetic to the Lie group determined by the Lie algebra given by*

$$(3.2) \quad \begin{aligned} [e_1, e_2] &= (4 \pm 2\sqrt{2})e_3, & [e_1, e_3] &= -e_2, & [e_2, e_3] &= (3 \pm 2\sqrt{2})e_1 \\ [e_4, e_1] &= -e_2, & [e_4, e_2] &= (3 \pm 2\sqrt{2})e_1, \end{aligned}$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis. Moreover, $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ does not admit any left-invariant $\check{\rho}$ -Einstein metric.

Proof. Following Milnor [7], let $\{v_1, v_2, v_3\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}_3$ ($\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{su}(2)$) such that

$$[v_2, v_3] = \lambda_1 v_1, \quad [v_3, v_1] = \lambda_2 v_2, \quad [v_1, v_2] = \lambda_3 v_3, \quad \lambda_1 \lambda_2 \lambda_3 \neq 0.$$

Here λ_1, λ_2 and λ_3 are all positive for the Lie algebra $\mathfrak{su}(2)$ and $\lambda_1 \lambda_2 \lambda_3 < 0$ in the $\mathfrak{sl}(2, \mathbb{R})$ case. Take v_4 (not necessarily orthogonal to $\mathfrak{g}_3$) so that $[v_4, v_i] = 0$, for all $i = 1, 2, 3$. Finally set $e_k = v_k$, ($k = 1, 2, 3$) and normalize the vector $v_4 - \sum_{i=1}^3 \langle v_4, v_i \rangle v_i$ so that $\{e_1, \dots, e_4\}$ is an orthonormal basis with brackets given by

$$(3.3) \quad \begin{aligned} [e_1, e_2] &= \lambda_3 e_3, & [e_2, e_3] &= \lambda_1 e_1, & [e_3, e_1] &= \lambda_2 e_2, \\ [e_1, e_4] &= \frac{1}{R} \{k_3 \lambda_2 e_2 - k_2 \lambda_3 e_3\}, & [e_2, e_4] &= \frac{1}{R} \{k_1 \lambda_3 e_3 - k_3 \lambda_1 e_1\}, \\ [e_3, e_4] &= \frac{1}{R} \{k_2 \lambda_1 e_1 - k_1 \lambda_2 e_2\}, & & & R > 0. \end{aligned}$$

Let $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ be the trace-free $\check{\rho}$ -tensor. A straightforward calculation shows that the $\check{\rho}_0$ components are given by

$$16R^4\check{\rho}_{011} = \mathfrak{C}_{11}, \quad 4R^4\check{\rho}_{012} = \mathfrak{C}_{12}, \quad 4R^4\check{\rho}_{013} = \mathfrak{C}_{13}, \quad 4R^3\check{\rho}_{014} = \mathfrak{C}_{14},$$

$$16R^4\check{\rho}_{022} = \mathfrak{C}_{22}, \quad 4R^4\check{\rho}_{023} = \mathfrak{C}_{23}, \quad 4R^3\check{\rho}_{024} = \mathfrak{C}_{24}, \quad 16R^4\check{\rho}_{033} = \mathfrak{C}_{33},$$

$$4R^3\check{\rho}_{034} = \mathfrak{C}_{34}, \quad 16R^4\check{\rho}_{044} = \mathfrak{C}_{44},$$

where the polynomials $\mathfrak{C}_{ij}$ are determined as follows.

$$\begin{aligned} \mathfrak{C}_{11} &= R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 - 10R^4\lambda_1^2\lambda_2^2 + 4k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 - 3k_1^4\lambda_2^4 - 2k_1^2R^2\lambda_2^4 + R^4\lambda_2^4 \\ &\quad + k_3^4(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2) + 4R^4\lambda_1^3\lambda_3 + 12R^4\lambda_1^2\lambda_2\lambda_3 - 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3 - 4R^4\lambda_1\lambda_2^2\lambda_3 \\ &\quad + 4k_1^4\lambda_2^3\lambda_3 - 8k_1^2R^2\lambda_2^3\lambda_3 - 12R^4\lambda_2^3\lambda_3 - 10R^4\lambda_1^2\lambda_2^3 - 4k_1^2R^2\lambda_1\lambda_2\lambda_2^2 - 4R^4\lambda_1\lambda_2\lambda_2^2 - 2k_1^4\lambda_2^2\lambda_3^2 \\ &\quad + 20k_1^2R^2\lambda_2^2\lambda_3^2 + 22R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3 + 4k_1^4\lambda_2\lambda_3^3 - 8k_1^2R^2\lambda_2\lambda_3^3 - 12R^4\lambda_2\lambda_3^3 \\ &\quad - 3k_1^4\lambda_3^4 - 2k_1^2R^2\lambda_3^4 + R^4\lambda_3^4 + k_2^4(\lambda_1 - \lambda_3)^2(\lambda_1^2 + 6\lambda_1\lambda_3 + \lambda_3^2) \\ &\quad + 2k_2^2(R^2(\lambda_1 - \lambda_3)(\lambda_1^3 + \lambda_1^2(2\lambda_2 + 5\lambda_3) + \lambda_1(-4\lambda_2^2 + 8\lambda_2\lambda_3 - 5\lambda_3^2)) \\ &\quad - \lambda_3(4\lambda_2^2 - 6\lambda_2\lambda_3 + \lambda_3^2)) - k_1^2(\lambda_1^2\lambda_2(\lambda_2 - 2\lambda_3) + \lambda_3^3(-2\lambda_2 + \lambda_3) \\ &\quad - 2\lambda_1\lambda_3(\lambda_2^2 - 3\lambda_2\lambda_3 + \lambda_3^2)) + k_3^2(\lambda_1^4 + \lambda_2^2\lambda_3^2 + 2\lambda_1^3(\lambda_2 + \lambda_3) + 2\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3) \\ &\quad - 2\lambda_1^2(2\lambda_2^2 + \lambda_2\lambda_3 + 2\lambda_3^2))) + 2k_3^2(k_1^2(-\lambda_2^3(\lambda_2 - 2\lambda_3) + \lambda_1^2(2\lambda_2 - \lambda_3)\lambda_3 \\ &\quad + 2\lambda_1\lambda_2(\lambda_2^2 - 3\lambda_2\lambda_3 + \lambda_3^2)) + R^2(\lambda_1 - \lambda_2)(\lambda_1^3 + \lambda_1^2(5\lambda_2 + 2\lambda_3) + \lambda_1(-5\lambda_2^2 + 8\lambda_2\lambda_3 - 4\lambda_3^2) \\ &\quad - \lambda_2(\lambda_2^2 - 6\lambda_2\lambda_3 + 4\lambda_3^2))), \\ \mathfrak{C}_{12} &= k_1k_2(R^2\lambda_1^2\lambda_2^2 - k_1^2\lambda_1\lambda_2^3 - 4R^2\lambda_1^2\lambda_2\lambda_3 - 4R^2\lambda_1\lambda_2^2\lambda_3 + R^2\lambda_1^2\lambda_3^2 + k_1^2\lambda_1\lambda_2\lambda_3^2 + 6R^2\lambda_1\lambda_2\lambda_2^2 \\ &\quad + k_1^2\lambda_2^2\lambda_3^2 + R^2\lambda_2^2\lambda_3^2 - k_1^2\lambda_3^4 - R^2\lambda_3^4 - k_2^2(\lambda_1^2 - \lambda_2^2)(\lambda_1\lambda_2 - \lambda_2^2) \\ &\quad + k_3^2(\lambda_1^2\lambda_2^2 - \lambda_1^3\lambda_3 - \lambda_2^3\lambda_3 + \lambda_1\lambda_2\lambda_3^2)), \\ \mathfrak{C}_{13} &= k_1k_3(R^2\lambda_1^2\lambda_2^2 - k_1^2\lambda_2^4 - R^2\lambda_2^4 - 4R^2\lambda_1^2\lambda_2\lambda_3 + k_1^2\lambda_1\lambda_2^2\lambda_3 + 6R^2\lambda_1\lambda_2^2\lambda_3 + R^2\lambda_1^2\lambda_3^2 \\ &\quad - 4R^2\lambda_1\lambda_2\lambda_3^2 + k_1^2\lambda_2^2\lambda_3^2 + R^2\lambda_2^2\lambda_3^2 - k_1^2\lambda_1\lambda_3^3 - k_3^2(\lambda_1^2 - \lambda_2^2)(-\lambda_2^2 + \lambda_1\lambda_3) \\ &\quad + k_2^2(-\lambda_1^3\lambda_2 + \lambda_1\lambda_2^2\lambda_3 + \lambda_1^2\lambda_3^2 - \lambda_2\lambda_3^2)), \\ \mathfrak{C}_{14} &= -k_1((R^2(-\lambda_1^2 + (\lambda_2 - \lambda_3)^2) + k_1^2(\lambda_2 - \lambda_3)^2)(\lambda_2 - \lambda_3)^2 \\ &\quad + k_2^2(\lambda_1 - \lambda_3)(\lambda_1^2\lambda_2 - \lambda_1\lambda_3(\lambda_2 + \lambda_3) - \lambda_3(2\lambda_2^2 - 4\lambda_2\lambda_3 + \lambda_3^2)) \\ &\quad + k_3^2(\lambda_1 - \lambda_2)(\lambda_1^2\lambda_3 - \lambda_1\lambda_2(\lambda_2 + \lambda_3) - \lambda_2(\lambda_2^2 - 4\lambda_2\lambda_3 + 2\lambda_3^2))), \\ \mathfrak{C}_{22} &= R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 - 8k_1^2R^2\lambda_1^2\lambda_2^2 - 10R^4\lambda_1^2\lambda_2^2 + 4k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 + k_1^4\lambda_2^4 \\ &\quad + 2k_1^2R^2\lambda_2^4 + R^4\lambda_2^4 + k_3^4(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2) - 12R^4\lambda_1^3\lambda_3 - 4R^4\lambda_1^2\lambda_2\lambda_3 + 12k_1^2R^2\lambda_1\lambda_2^2\lambda_3 \\ &\quad + 12R^4\lambda_1\lambda_2^2\lambda_3 + 4k_1^4\lambda_2^3\lambda_3 + 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 + 8k_1^2R^2\lambda_1^2\lambda_2^3 + 22R^4\lambda_1^2\lambda_2^3 - 4k_1^2R^2\lambda_1\lambda_2\lambda_2^2 \\ &\quad - 4R^4\lambda_1\lambda_2\lambda_2^2 - 10k_1^4\lambda_2^2\lambda_3^2 - 20k_1^2R^2\lambda_2^2\lambda_3^2 - 10R^4\lambda_2^2\lambda_3^2 - 12k_1^2R^2\lambda_1\lambda_3^3 - 12R^4\lambda_1\lambda_3^3 + 4k_1^4\lambda_2\lambda_3^3 \\ &\quad + 8k_1^2R^2\lambda_2\lambda_3^3 + 4R^4\lambda_2\lambda_3^3 + k_1^4\lambda_3^4 + 2k_1^2R^2\lambda_3^4 + R^4\lambda_3^4 - k_2^4(\lambda_1 - \lambda_3)^2(3\lambda_1^2 + 2\lambda_1\lambda_3 + 3\lambda_3^2) \\ &\quad - 2k_2^2(R^2(\lambda_1 - \lambda_3)^2(\lambda_1^2 - 2\lambda_1\lambda_2 + 6\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2) \\ &\quad + k_3^2(\lambda_1^4 + 6\lambda_1^2\lambda_2\lambda_3 + \lambda_2^2\lambda_3^2 - 2\lambda_1^3(\lambda_2 + \lambda_3) - 2\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3)) + k_1^2(\lambda_1^2\lambda_2(\lambda_2 - 2\lambda_3) \\ &\quad + \lambda_3^3(-2\lambda_2 + \lambda_3) - 2\lambda_1\lambda_3(\lambda_2^2 - 3\lambda_2\lambda_3 + \lambda_3^2))) + 2k_3^2(k_1^2(\lambda_2^2(\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2) \\ &\quad + 2\lambda_1\lambda_2(\lambda_2^2 - \lambda_2\lambda_3 + \lambda_3^2) + \lambda_1^2(-4\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_3^2)) \\ &\quad + R^2(\lambda_1 - \lambda_2)(\lambda_1^3 + \lambda_1^2(5\lambda_2 - 6\lambda_3) - \lambda_2(\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2) + \lambda_1(-5\lambda_2^2 - 8\lambda_2\lambda_3 + 4\lambda_3^2))), \\ \mathfrak{C}_{23} &= k_2k_3(-R^2\lambda_1^4 + R^2\lambda_1^2\lambda_2^2 - k_1^2\lambda_1\lambda_2^3 + k_1^2\lambda_1^2\lambda_2\lambda_3 + 6R^2\lambda_1^2\lambda_2\lambda_3 - 4R^2\lambda_1\lambda_2^2\lambda_3 + R^2\lambda_1^2\lambda_3^2 \\ &\quad - 4R^2\lambda_1\lambda_2\lambda_3^2 + k_1^2\lambda_2^2\lambda_3^2 + R^2\lambda_2^2\lambda_3^2 - k_1^2\lambda_1\lambda_3^3 - k_3^2(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_2\lambda_3) - k_2^2(\lambda_1^2 - \lambda_2\lambda_3)(\lambda_1^2 - \lambda_3^2)), \\ \mathfrak{C}_{24} &= -k_2(R^2\lambda_1^4 - R^2\lambda_1^2\lambda_2^2 + k_1^2\lambda_1\lambda_2^3 + k_2^2(\lambda_1 - \lambda_3)^4 - 4R^2\lambda_1^3\lambda_3 - 2k_1^2\lambda_1^2\lambda_2\lambda_3 - 2k_1^2\lambda_1\lambda_2^2\lambda_3 \\ &\quad + 2R^2\lambda_1\lambda_2^2\lambda_3 + 2k_1^2\lambda_1^2\lambda_3^2 + 6R^2\lambda_1^2\lambda_3^2 + 5k_1^2\lambda_1\lambda_2\lambda_3^2 - k_1^2\lambda_2^2\lambda_3^2 - R^2\lambda_2^2\lambda_3^2 - 4k_1^2\lambda_1\lambda_3^3 - 4R^2\lambda_1\lambda_3^3 \\ &\quad + k_1^4\lambda_3^4 + R^2\lambda_3^4 + k_3^2(\lambda_1 - \lambda_2)(\lambda_1^3 + \lambda_1^2(\lambda_2 - 4\lambda_3) - \lambda_2^2\lambda_3 + \lambda_1\lambda_3(\lambda_2 + 2\lambda_3))), \\ \mathfrak{C}_{33} &= R^4\lambda_1^4 - 12R^4\lambda_1^3\lambda_2 + 8k_1^2R^2\lambda_1^2\lambda_2^2 + 22R^4\lambda_1^2\lambda_2^2 - 12k_1^2R^2\lambda_1\lambda_2^3 - 12R^4\lambda_1\lambda_2^3 + k_1^4\lambda_2^4 \\ &\quad + 2k_1^2R^2\lambda_2^4 + R^4\lambda_2^4 - k_3^4(\lambda_1 - \lambda_2)^2(3\lambda_1^2 + 2\lambda_1\lambda_2 + 3\lambda_2^2) + 4R^4\lambda_1^3\lambda_3 - 4R^4\lambda_1^2\lambda_2\lambda_3 - 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3 \end{aligned}$$

$$\begin{aligned}
& -4R^4\lambda_1\lambda_2\lambda_3 + 4k_1^4\lambda_2^3\lambda_3 + 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 - 8k_1^2R^2\lambda_1^2\lambda_3^2 - 10R^4\lambda_1^2\lambda_3^2 + 12k_1^2R^2\lambda_1\lambda_2\lambda_3^2 \\
& + 12R^4\lambda_1\lambda_2\lambda_3^2 - 10k_1^4\lambda_2^2\lambda_3^2 - 20k_1^2R^2\lambda_2^2\lambda_3^2 - 10R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3 + 4k_1^4\lambda_2\lambda_3^3 \\
& + 8k_1^2R^2\lambda_2\lambda_3^3 + 4R^4\lambda_2\lambda_3^3 + k_1^4\lambda_3^4 + 2k_1^2R^2\lambda_3^4 + R^4\lambda_3^4 \\
& + k_2^4(\lambda_1 - \lambda_3)^2(\lambda_1^2 + 6\lambda_1\lambda_3 + \lambda_3^2) - 2k_2^3(R^2(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3) \\
& + k_1^2(\lambda_2^3(\lambda_2 - 2\lambda_3) + \lambda_1^2\lambda_3(-2\lambda_2 + \lambda_3) - 2\lambda_1\lambda_2(\lambda_2^2 - 3\lambda_2\lambda_3 + \lambda_3^2))) \\
& + 2k_2^2(-k_3^2(\lambda_1^4 + 6\lambda_1^2\lambda_2\lambda_3 + \lambda_2^2\lambda_3^2 - 2\lambda_1^3(\lambda_2 + \lambda_3) - 2\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3)) \\
& + R^2(\lambda_1 - \lambda_3)(\lambda_1^3 + \lambda_1^2(-6\lambda_2 + 5\lambda_3) + \lambda_1(4\lambda_2^2 - 8\lambda_2\lambda_3 - 5\lambda_3^2) - \lambda_3(-4\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_3^2)) \\
& + k_1^2(\lambda_1^2(\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2) + 2\lambda_1\lambda_3(\lambda_2^2 - \lambda_2\lambda_3 + \lambda_3^2) + \lambda_3^2(-4\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_3^2))), \\
\mathfrak{C}_{34} = & -k_3(R^2\lambda_1^4 + k_3^2(\lambda_1 - \lambda_2)^4 - 4R^2\lambda_1^3\lambda_2 + 2k_1^2\lambda_1^2\lambda_2^2 + 6R^2\lambda_1^2\lambda_2^2 - 4k_1^2\lambda_1\lambda_2^3 - 4R^2\lambda_1\lambda_2^3 \\
& + k_1^2\lambda_2^4 + R^2\lambda_2^4 - 2k_1^2\lambda_1^2\lambda_2\lambda_3 + 5k_1^2\lambda_1\lambda_2^2\lambda_3 - R^2\lambda_1^2\lambda_3^2 - 2k_1^2\lambda_1\lambda_2\lambda_3^2 + 2R^2\lambda_1\lambda_2\lambda_3^2 - k_1^2\lambda_2^2\lambda_3^2 \\
& - R^2\lambda_2^2\lambda_3^2 + k_1^2\lambda_1\lambda_3^3 + k_2^2(\lambda_1 - \lambda_3)(\lambda_1^3 - \lambda_2\lambda_3^2 + \lambda_1^2(-4\lambda_2 + \lambda_3) + \lambda_1\lambda_2(2\lambda_2 + \lambda_3))), \\
\mathfrak{C}_{44} = & -3R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 - 2R^4\lambda_1^2\lambda_2^2 + 4k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 + k_1^4\lambda_2^4 - 2k_1^2R^2\lambda_2^4 - 3R^4\lambda_2^4 \\
& + k_3^4(\lambda_1 - \lambda_2)^2(\lambda_1^2 - 10\lambda_1\lambda_2 + \lambda_2^2) + 4R^4\lambda_1^3\lambda_3 - 4R^4\lambda_1^2\lambda_2\lambda_3 - 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3 - 4R^4\lambda_1\lambda_2^2\lambda_3 \\
& - 12k_1^4\lambda_2^3\lambda_3 - 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 - 2R^4\lambda_1^2\lambda_3^2 - 4k_1^2R^2\lambda_1\lambda_2\lambda_3^2 - 4R^4\lambda_1\lambda_2\lambda_3^2 + 22k_1^4\lambda_2^2\lambda_3^2 \\
& + 20k_1^2R^2\lambda_2^2\lambda_3^2 - 2R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3 - 12k_1^4\lambda_2\lambda_3^3 - 8k_1^2R^2\lambda_2\lambda_3^3 + 4R^4\lambda_2\lambda_3^3 \\
& + k_1^4\lambda_3^4 - 2k_1^2R^2\lambda_3^4 - 3R^4\lambda_3^4 + k_2^4(\lambda_1 - \lambda_3)^2(\lambda_1^2 - 10\lambda_1\lambda_3 + \lambda_3^2) \\
& + 2k_2^2(-R^2(\lambda_1 - \lambda_3)^2(\lambda_1^2 - 2\lambda_1\lambda_2 + 6\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2) + k_3^2(\lambda_1^4 + \lambda_2^2\lambda_3^2 - 6\lambda_1^3(\lambda_2 + \lambda_3) \\
& - 6\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3) + 2\lambda_1^2(2\lambda_2^2 + 7\lambda_2\lambda_3 + 2\lambda_3^2)) + k_1^2(\lambda_3^2(4\lambda_2^2 - 6\lambda_2\lambda_3 + \lambda_3^2) \\
& - 2\lambda_1\lambda_3(3\lambda_2^2 - 7\lambda_2\lambda_3 + 3\lambda_3^2) + \lambda_1^2(\lambda_2^2 - 6\lambda_2\lambda_3 + 4\lambda_3^2))) \\
& - 2k_2^3(R^2(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3) - k_1^2(\lambda_1^2(4\lambda_2^2 - 6\lambda_2\lambda_3 + \lambda_3^2) \\
& - 2\lambda_1\lambda_2(3\lambda_2^2 - 7\lambda_2\lambda_3 + 3\lambda_3^2) + \lambda_2^2(\lambda_2^2 - 6\lambda_2\lambda_3 + 4\lambda_3^2))).
\end{aligned}$$

Hence a left-invariant metric is $\check{\rho}$ -Einstein if and only if the system $\{\mathfrak{C}_{ij} = 0\}$ has a real solution. First of all, observe that if $\lambda_1 = \lambda_2 = \lambda_3$, then the Ricci operator takes the form $\text{Ric} = \frac{1}{2} \text{diag}[\lambda_1^2, \lambda_1^2, \lambda_1^2, 0]$, which shows that it cannot be $\check{\rho}$ -Einstein.

Now, we consider the terms k_i at (3.3). Up to a permutation of the basis $\{e_1, e_2, e_3\}$ one may assume that one of the following holds:

$$(i) \ k_1 k_2 k_3 \neq 0, \quad (ii) \ k_1 = 0, k_2 k_3 \neq 0, \quad (iii) \ k_1 = k_2 = 0, k_3 \neq 0.$$

We analyze the different possibilities separately.

3.1.1. (i) $k_1 k_2 k_3 \neq 0$. Let $\mathcal{I} \subset \mathbb{R}[k_1, k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ be the ideal generated by the polynomials $\mathfrak{C}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and a detailed analysis of that basis (using that $k_1 k_2 k_3 \neq 0$) shows that the polynomial

$$R^4\lambda_1\lambda_2^2\lambda_3^2(\lambda_2^4 - \lambda_2^3\lambda_3 + \lambda_2\lambda_3^3 - \lambda_3^4)$$

belongs to $\mathcal{G}$. Since $R^4\lambda_1\lambda_2^2\lambda_3^2 \neq 0$, one has that $\lambda_3 = \pm\lambda_2$.

Next, we compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{\lambda_3 + \lambda_2\}$ with respect to the lexicographical order so that the polynomial $R^4\lambda_1^2\lambda_3^3$ belongs to $\mathcal{G}_1$, which gives a contradiction since $R^4\lambda_1^2\lambda_3^3 \neq 0$. Proceeding in the same way, let $\mathcal{G}_2$ of the ideal generated by $\mathcal{G} \cup \{\lambda_3 - \lambda_2\}$ with respect to the lexicographical order. Then $R^6\lambda_1\lambda_3^4$ belongs to $\mathcal{G}_2$, thus showing that no $\check{\rho}$ -Einstein left-invariant metrics may exist.

3.1.2. (ii) $k_1 = 0, k_2 k_3 \neq 0$. We compute a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials $\{\mathfrak{C}_{ij}\} \subset \mathbb{R}[k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ with respect to the lexicographical order so that $\mathbf{g}_1 = R^8\lambda_2\lambda_3^3(10\lambda_2^2 + 19\lambda_2\lambda_3 + 10\lambda_3^2) \in \mathcal{G}$. Since $R\lambda_2\lambda_3 \neq 0$

and $10\lambda_2^2 + 19\lambda_2\lambda_3 + 10\lambda_3^2 = 0$ has no real solutions, we get that no $\check{\rho}$ -Einstein left-invariant metric may exist.

3.1.3. (iii) $k_1 = k_2 = 0$, $k_3 \neq 0$. We compute a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials $\{\mathfrak{C}_{ij}\} \subset \mathbb{R}[\lambda_1, \lambda_2, \lambda_3, k_3, R]$ with respect to the lexicographical order so that $\mathbf{g}_1 = k_3^2 R^8 (k_3^2 - R^2) \lambda_3^7 \in \mathcal{G}$. Since $k_3 R \lambda_3 \neq 0$, the only possible solutions of $\mathbf{g}_1 = 0$ are $k_3 = \pm R$.

Assume $k_3 = R$ and compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{k_3 - R\}$. Since the polynomial $\mathbf{g}_2 = R^4(\lambda_1 + \lambda_2 - \lambda_3)\lambda_3^5 \in \mathcal{G}_1$, one has that $\lambda_3 = \lambda_1 + \lambda_2$. Proceeding exactly in the same way under the assumption $k_3 = -R$, one gets as in the previous case that $\lambda_3 = \lambda_1 + \lambda_2$. Now a straightforward calculation shows that

$$\mathfrak{C}_{34} = -R^3(\lambda_1 - \lambda_2)^2(\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2).$$

If $\lambda_1 = \lambda_2$, then $\mathfrak{C}_{11} = -16R^4\lambda_2^4 \neq 0$. Hence assume $\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2 = 0$, and thus $\lambda_1 = (3 \pm 2\sqrt{2})\lambda_2$. A straightforward calculation now shows that the Ricci operator takes the form

$$\text{Ric} = 2\lambda_2^2 \begin{pmatrix} 4 \pm 3\sqrt{2} & 0 & 0 & 0 \\ 0 & -4 \mp 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 3 \pm 2\sqrt{2} & 3 \pm 2\sqrt{2} \\ 0 & 0 & 3 \pm 2\sqrt{2} & -3 \mp 2\sqrt{2} \end{pmatrix},$$

from where it follows that the left-invariant metric determined by $k_3 = R$, $\lambda_1 = \lambda_3 - \lambda_2$, $\lambda_3 = (4 + 2\sqrt{2})\lambda_2$ is $\check{\rho}$ -Einstein. Finally, considering the homothetic basis $e_k = \frac{1}{\lambda_2}e_k$, the Lie algebra structure (3.2) is obtained, thus completing the proof. $\square$

3.2. Left-invariant $\check{\rho}$ -Einstein metrics in $\mathbb{R}e_4 \times E(1, 1)$ and $\mathbb{R}e_4 \times E(2)$.

Lemma 4. $\mathbb{R}e_4 \times E(1, 1)$ and $\mathbb{R}e_4 \times E(2)$ do not admit non-Einstein $\check{\rho}$ -Einstein left invariant metrics.

Proof. Let $\mathfrak{g}_3$ be either the Poincare algebra $\mathfrak{e}(1, 1)$ or the Euclidean algebra $\mathfrak{e}(2)$. There exist an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathfrak{g}_3$ such that (see [7])

$$[v_1, v_2] = 0, \quad [v_2, v_3] = \lambda_1 v_1, \quad [v_3, v_1] = \lambda_2 v_2,$$

where $\lambda_1\lambda_2 \neq 0$. Moreover $\lambda_1\lambda_2 > 0$ if $\mathfrak{g}_3 = E(2)$ ($\lambda_1\lambda_2 < 0$ if $\mathfrak{g}_3 = E(1, 1)$). Expressing a derivation of $\mathfrak{g}_3$ with respect to $\{v_1, v_2, v_3\}$, one has

$$\text{der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} b & a & c \\ -\frac{\lambda_2}{\lambda_1}a & b & d \\ 0 & 0 & 0 \end{pmatrix}; a, b, c, d \in R \right\}.$$

Extend $\{v_1, v_2, v_3\}$ to a basis $\{v_1, v_2, v_3, v_4\}$ so that $\text{ad}(v_4)$ is determined by the derivation and consider an orthonormal basis $\{e_i\}$ so that $e_i = v_i$ for $i = 1, 2, 3$ and e_4 is obtained after normalization of $v_4 - \sum_i \langle v_4, v_i \rangle v_i$. Then one has the nonzero Lie brackets

$$(3.4) \quad \begin{aligned} [e_2, e_3] &= \lambda_1 e_1, & [e_3, e_1] &= \lambda_2 e_2, \\ [e_4, e_1] &= \frac{1}{R} \{ b e_1 - \lambda_2 \left(\frac{a}{\lambda_1} + k_3 \right) e_2 \}, & [e_4, e_2] &= \frac{1}{R} \{ (a + k_3 \lambda_1) e_1 + b e_2 \}, \\ [e_4, e_3] &= \frac{1}{R} \{ (c - k_2 \lambda_1) e_1 + (d + k_1 \lambda_2) e_2 \}, & R &> 0, \end{aligned}$$

To simplify the notation, set $A = \frac{a}{\lambda_1} + k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$.

Moreover, since $\lambda_1\lambda_2 \neq 0$, we work with an homothetic basis $\hat{e}_i = \frac{1}{\lambda_1}e_i$, so that we may assume $\lambda_1 = 1$.

A straightforward calculation shows that the components of $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ are given by

$$16R^4\check{\rho}_{011} = \mathfrak{C}_{11}, \quad 4R^4\check{\rho}_{012} = \mathfrak{C}_{12}, \quad 4R^4\check{\rho}_{013} = \mathfrak{C}_{13}, \quad 4R^3\check{\rho}_{014} = \mathfrak{C}_{14},$$

$$16R^4\check{\rho}_{022} = \mathfrak{C}_{22}, \quad 4R^4\check{\rho}_{023} = \mathfrak{C}_{23}, \quad 4R^3\check{\rho}_{024} = \mathfrak{C}_{24}, \quad 16R^4\check{\rho}_{033} = \mathfrak{C}_{33},$$

$$4R^3\check{\rho}_{034} = \mathfrak{C}_{34}, \quad 16R^4\check{\rho}_{044} = \mathfrak{C}_{44},$$

where the polynomials $\mathfrak{C}_{ij}$ now become

$$\begin{aligned} \mathfrak{C}_{11} &= 16b^4 + C^4 - 2C^2D^2 - 3D^4 + 2C^2R^2 + R^4 + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 \\ &\quad + 4R^4\lambda_2 - 8C^2R^2\lambda_2^2 - 2D^2R^2\lambda_2^2 - 10R^4\lambda_2^2 + 4R^4\lambda_2^3 + R^4\lambda_2^4 - 4AbCD(5 + \lambda_2) \\ &\quad + A^4(-1 + \lambda_2)^2(1 + 6\lambda_2 + \lambda_2^2) + 2A^2(R^2 + 2D^2\lambda_2 + 4R^2\lambda_2 - D^2\lambda_2^2 - 10R^2\lambda_2^2 \\ &\quad + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(1 + 2\lambda_2 - 4\lambda_2^2) + 16b^2(-1 + \lambda_2^2)) \\ &\quad - 2b^2(7C^2 + 9D^2 - 16R^2(-1 + \lambda_2^2)), \\ \mathfrak{C}_{12} &= -A^2CD\lambda_2 + CD(b^2 + C^2 + D^2 - R^2\lambda_2) + Ab(-16b^2(-1 + \lambda_2) \\ &\quad - D^2(2 + \lambda_2) + C^2(1 + 2\lambda_2)), \\ \mathfrak{C}_{13} &= 6bC(2b^2 + R^2(-1 + \lambda_2)\lambda_2) + A^2bC(-1 - 2\lambda_2 + 3\lambda_2^2) + A^3D(\lambda_2 - \lambda_2^3) \\ &\quad - AD(C^2 + 2b^2(-3 + 5\lambda_2) + \lambda_2(D^2 + R^2(-1 + \lambda_2^2))), \\ \mathfrak{C}_{14} &= -(A^2D\lambda_2(-1 + \lambda_2^2) + AbC(1 - 4\lambda_2 + 3\lambda_2^2) + D(C^2 + \lambda_2(8b^2 + D^2 \\ &\quad + R^2(-1 + \lambda_2^2)))), \\ \mathfrak{C}_{22} &= 16b^4 - 3C^4 - 2C^2D^2 + D^4 - 2C^2R^2 - 8D^2R^2 + R^4 + 4C^2R^2\lambda_2 \\ &\quad + 4D^2R^2\lambda_2 + 4R^4\lambda_2 + 2D^2R^2\lambda_2^2 - 10R^4\lambda_2^2 + 4R^4\lambda_2^3 + R^4\lambda_2^4 + 4AbCD(1 + 5\lambda_2) \\ &\quad + A^4(-1 + \lambda_2)^2(1 + 6\lambda_2 + \lambda_2^2) + 2A^2(-4D^2 + R^2 + 2D^2\lambda_2 + 4R^2\lambda_2 + D^2\lambda_2^2 \\ &\quad - 10R^2\lambda_2^2 + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(-1 + 2\lambda_2) - 16b^2(-1 + \lambda_2^2)) \\ &\quad - 2b^2(9C^2 + 7D^2 + 16R^2(-1 + \lambda_2^2)), \\ \mathfrak{C}_{23} &= 6bD(2b^2 - R^2(-1 + \lambda_2)) - A^2bD(-3 + 2\lambda_2 + \lambda_2^2) + A^3(C - C\lambda_2^2) \\ &\quad + AC(C^2 + R^2 + b^2(10 - 6\lambda_2) + D^2\lambda_2 - R^2\lambda_2^2), \\ \mathfrak{C}_{24} &= -AbD(3 - 4\lambda_2 + \lambda_2^2) + A^2(C - C\lambda_2^2) + C(8b^2 + C^2 + R^2 + D^2\lambda_2 - R^2\lambda_2^2), \\ \mathfrak{C}_{33} &= -48b^4 + C^4 + 2C^2D^2 + D^4 + 18b^2(C^2 + D^2) + 2C^2R^2 + 8D^2R^2 + R^4 \\ &\quad - 20AbCD(-1 + \lambda_2) - 12C^2R^2\lambda_2 - 12D^2R^2\lambda_2 - 12R^4\lambda_2 + 8C^2R^2\lambda_2^2 + 2D^2R^2\lambda_2^2 \\ &\quad + 22R^4\lambda_2^2 - 12R^4\lambda_2^3 + R^4\lambda_2^4 - A^4(-1 + \lambda_2)^2(3 + 2\lambda_2 + 3\lambda_2^2) \\ &\quad - 2A^2(C^2 + R^2 + 8b^2(-1 + \lambda_2)^2 - 2C^2\lambda_2 - 2D^2\lambda_2 + 4R^2\lambda_2 + D^2\lambda_2^2 \\ &\quad - 10R^2\lambda_2^2 + 4R^2\lambda_2^3 + R^2\lambda_2^4), \\ \mathfrak{C}_{34} &= -(3bCD(-1 + \lambda_2) + A^3(-1 + \lambda_2)^4 + A(2D^2 + R^2 + 4b^2(-1 + \lambda_2)^2 - 4D^2\lambda_2 \\ &\quad - 4R^2\lambda_2 + D^2\lambda_2^2 + 6R^2\lambda_2^2 - 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(1 - 4\lambda_2 + 2\lambda_2^2))), \\ \mathfrak{C}_{44} &= 16b^4 + C^4 + 2C^2D^2 + D^4 + 14b^2(C^2 + D^2) - 2C^2R^2 - 3R^4 \\ &\quad + 4AbCD(-1 + \lambda_2) + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 + 4R^4\lambda_2 - 2D^2R^2\lambda_2^2 - 2R^4\lambda_2^2 + 4R^4\lambda_2^3 \\ &\quad - 3R^4\lambda_2^4 + A^4(-1 + \lambda_2)^2(1 - 10\lambda_2 + \lambda_2^2) + 2A^2(4D^2 - R^2 + 8b^2(-1 + \lambda_2)^2 - 6D^2\lambda_2 \\ &\quad - 4R^2\lambda_2 + D^2\lambda_2^2 + 10R^2\lambda_2^2 - 4R^2\lambda_2^3 - R^2\lambda_2^4 + C^2(1 - 6\lambda_2 + 4\lambda_2^2)). \end{aligned}$$

Hence a left-invariant metric determined by (3.4) is $\check{\rho}$ -Einstein if and only if the system of equations $\{\mathfrak{C}_{ij} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[A, b, \lambda_2, C, D, R]$ the ideal generated by the polynomials $\mathfrak{C}_{ij}$. We compute a Gröbner basis $\mathcal{G} \subset \mathcal{I}$ with respect to the lexicographic order so that the polynomial $\mathbf{g}_0 = CD^2(C^2 + D^2)R^4(5C^2 + D^2 + R^2)$ belongs to $\mathcal{G}$. Hence $C = 0$ or $D = 0$, since $R > 0$.

3.2.1. *Case $C = 0$.* Assuming $C = 0$, we compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{C\}$ with respect to the lexicographical order. Since $\mathbf{g}_1 = AD^4$ belongs to $\mathcal{G}_1$, one has: we used one has several time and if these sentences remain we must change does with appropriate expression that either $A = 0$ or $D = 0$.

If we have $C = D = 0$, then $\mathfrak{C}_{12} = -(16Ab^3(-1 + \lambda_2))$. Hence one has that $A = 0$, $b = 0$ or $\lambda_2 = 1$. If $A = 0$, then we have that

$$\begin{aligned}\mathfrak{C}_{11} &= 16b^4 + 32b^2R^2(\lambda_2^2 - 1) + R^4(\lambda_2 - 1)^2(1 + 6\lambda_2 + \lambda_2^2) \\ \mathfrak{C}_{22} &= 16b^4 - 32b^2R^2(\lambda_2^2 - 1) + R^4(\lambda_2^2 - 1)^2(1 + 6\lambda_2 + \lambda_2^2) \\ \mathfrak{C}_{33} &= -48b^4 + R^4(\lambda_2^2 - 1)^2(1 - 10\lambda_2 + \lambda_2^2) \\ \mathfrak{C}_{44} &= 16b^4 - R^4(\lambda_2^2 - 1)^2(3 + 2\lambda_2 + 3\lambda_2^2).\end{aligned}$$

Now one has $b = \pm R$ and $\lambda_2 = 1$, leading to an Einstein metric. Assume now that $A \neq 0$ and $b = 0$. Then $\mathfrak{C}_{34} = -(A(A^2 + R^2)(-1 + \lambda_2)^4)$ and one has that $\lambda_2 = 1$, leading again to an Einstein metric. Finally, assuming $A \neq 0$, $b \neq 0$ and $\lambda_2 = 1$, one has $\mathfrak{C}_{11} = b^4$ which has no solution since $b \neq 0$.

Next assume $C = A = 0$, $D \neq 0$. Then $\mathfrak{C}_{23} = 6bD(2b^2 - R^2(-1 + \lambda_2))$ shows that either $b = 0$ or $\lambda_2 = 1 + \frac{2b^2}{R^2}$. If $b = 0$ then $\mathfrak{C}_{14} = -D\lambda_2(D^2 + R^2(-1 + \lambda_2^2))$ and thus $D = \pm\sqrt{R^2(1 - \lambda_2^2)}$. Now a straightforward calculation gives

$$\begin{aligned}\mathfrak{C}_{11} &= \mathfrak{C}_{44} = 2R^4(-1 + 4\lambda_2 - 3\lambda_2^2) \\ \mathfrak{C}_{22} &= 2R^4(-3 + 4\lambda_2 - \lambda_2^2) \\ \mathfrak{C}_{33} &= 2R^4(5 - 12\lambda_2 + 7\lambda_2^2).\end{aligned}$$

Then one has that $\lambda_2 = 1$ and thus $D = 0$, which is a contradiction.

Assume now $\lambda_2 = 1 + \frac{2b^2}{R^2}$ and $b \neq 0$. Then $\mathfrak{C}_{14} = -\frac{1}{R^4}D(2b^2 + R^2)(4b^4 + 12b^2R^2 + D^2R^2)$. Since $\mathfrak{C}_{14} = 0$ has no real solutions, no $\check{\rho}$ -Einstein metrics may occur in this setting.

3.2.2. *Case $D = 0$.* Assume $C \neq 0$ and compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated $\mathcal{G} \cup \{D\}$ with respect to the lexicographic order. Since the polynomial $\mathbf{g}_2 = AC^4\lambda_2^2$ belongs to $\mathcal{G}_2$, one has that $A = 0$.

A standard calculation now gives $\mathfrak{C}_{13} = 6bC(2b^2 + R^2(-1 + \lambda_2)\lambda_2)$, and so either $b = 0$ or $b = \pm\frac{1}{\sqrt{2}}\sqrt{R^2(1 - \lambda_2)}\lambda_2$. If $b = 0$, then $\mathfrak{C}_{24} = C(C^2 - R^2(-1 + \lambda_2^2))$, and thus $C = \pm R\sqrt{\lambda_2^2 - 1}$. Then one has

$$\begin{aligned}\mathfrak{C}_{11} &= -2\lambda_2^2(1 - 4\lambda_2 + 3\lambda_2^2) \\ \mathfrak{C}_{22} &= \mathfrak{C}_{44} = -2\lambda_2^2(3 - 4\lambda_2 + \lambda_2^2) \\ \mathfrak{C}_{33} &= 2\lambda_2^2(7 - 12\lambda_2 + 5\lambda_2^2).\end{aligned}$$

Therefore one has $\lambda_2 = 1$ and the metric is Einstein. (IN THIS CASE, IF $\lambda_2 = 1$, THEN $C = 0$, WHICH IS A CONTRADICTION)

Setting $b = \pm\frac{1}{\sqrt{2}}\sqrt{R^2(1 - \lambda_2)}\lambda_2$, one has $\mathfrak{C}_{24} = C(C^2 + R^2(1 + 4\lambda_2 - 5\lambda_2^2))$. Hence $C = \pm R\sqrt{-1 - 4\lambda_2 + 5\lambda_2^2}$ and one has $\mathfrak{C}_{ii} = \pm R^4\lambda_2(-3 + 13\lambda_2 - 13\lambda_2^2 + 3\lambda_2^3)$, for all $i = 1, 2, 3, 4$. Since $\lambda_2 \neq 0$, the solutions of $\mathfrak{C}_{ii} = 0$ are $\lambda_2 = 1$, $\lambda_2 = 3$ and

$\lambda_2 = \frac{1}{3}$. The solution $\lambda_2 = 1$ gives rise to an Einstein metric, whereas $\lambda_2 = 3$ and $\lambda_2 = \frac{1}{3}$ give rise to some equation without real solutions. This completes the proof. $\square$

3.3. Left-invariant $\check{\rho}$ -Einstein metrics in $\mathbb{R}e_4 \ltimes H^3$.

Lemma 5. $\mathbb{R}e_4 \ltimes H^3$ admits a non-Einstein $\check{\rho}$ -Einstein left invariant metric if and only if it is homothetic to the Lie group determined by the Lie algebra

$$(3.5) \quad [e_1, e_2] = e_3, \quad [e_4, e_1] = -\frac{1}{2}e_1, \quad [e_4, e_2] = \frac{1}{2}e_3,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

Proof. Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{h}_3$ be a semi-direct product of $\mathbb{R}$ with the Heisenberg algebra $\mathfrak{h}_3$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $\mathfrak{h}_3$ so that

$$[v_1, v_2] = \gamma v_3, \quad [v_2, v_3] = 0, \quad [v_1, v_3] = 0, \quad \gamma \neq 0.$$

The derivations of $\mathfrak{h}_3$ with respect to the basis $\{v_1, v_2, v_3\}$ are given by

$$\text{der}(\mathfrak{h}_3) = \left\{ \begin{pmatrix} a & c & 0 \\ -c & d & 0 \\ h & f & a+d \end{pmatrix}; a, c, d, h, f \in R \right\}.$$

Let $\{v_1, v_2, v_3, v_4\}$ be a basis of $\mathfrak{g}$, where $\text{ad}(e_4)$ is determined by a derivation as above. After normalization, as in the previous cases, there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ where the nonzero Lie brackets are given as follows

$$(3.6) \quad \begin{aligned} [e_1, e_2] &= \gamma e_3, & [e_4, e_1] &= \frac{1}{R}\{ae_1 - ce_2 + (h + k_2\gamma)e_3\}, \\ [e_4, e_3] &= \frac{1}{R}(a+d)e_3, & [e_4, e_2] &= \frac{1}{R}\{ce_1 + de_2 + (f - k_1\gamma)e_3\}, \quad R > 0. \end{aligned}$$

In order to simplify the notation, set $F = f - k_1\gamma$ and $H = h + k_2\gamma$. Moreover, since $\gamma \neq 0$, working with an homothetic basis $\hat{e}_i = \frac{1}{\gamma}e_i$, so that we may assume $\gamma = 1$. Then the components of the tracefree tensor $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ are given by

$$\begin{aligned} 16R^4\check{\rho}_{011} &= \mathfrak{C}_{11}, & 4R^4\check{\rho}_{012} &= \mathfrak{C}_{12}, & 4R^4\check{\rho}_{013} &= \mathfrak{C}_{13}, & 4R^3\check{\rho}_{014} &= \mathfrak{C}_{14}, \\ 16R^4\check{\rho}_{022} &= \mathfrak{C}_{22}, & 4R^4\check{\rho}_{023} &= \mathfrak{C}_{23}, & 4R^3\check{\rho}_{024} &= \mathfrak{C}_{24}, & 16R^4\check{\rho}_{033} &= \mathfrak{C}_{33}, \\ 4R^3\check{\rho}_{034} &= \mathfrak{C}_{34}, & 16R^4\check{\rho}_{044} &= \mathfrak{C}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{C}_{ij}$ are as follows

$$\begin{aligned} \mathfrak{C}_{11} &= 16a^4 - 48d^4 - 16d^2F^2 - 3F^4 - 28cdFH + 18d^2H^2 - 2F^2H^2 + H^4 \\ &\quad + 2c^2(4d^2 + F^2 - H^2) - 2F^2R^2 + 2H^2R^2 + R^4 + 2a^2(4c^2 - 56d^2 - 9F^2 + 16H^2 + 16R^2) \\ &\quad - 4a(4c^2d + 3cFH + 2d(16d^2 + 3F^2 - 7H^2 - 4R^2)), \\ \mathfrak{C}_{12} &= 8a^3c - c^2FH + 2a^2(4cd + 5FH) + cd(-8d^2 - 4F^2 + H^2 - 4R^2) \\ &\quad + FH(10d^2 + F^2 + H^2 + R^2) + a(21dFH - c(8d^2 + F^2 - 4(H^2 + R^2))), \\ \mathfrak{C}_{13} &= 16a^3H - 2c^2dH + d(12d^2 - F^2)H + a^2(-2cF + 48dH) + a(-14cdF + 2c^2H \\ &\quad + (44d^2 + F^2)H) + cF(-8d^2 + F^2 + H^2), \\ \mathfrak{C}_{14} &= 8a^2F + 8adF + 4d^2F + F^3 - 2acH + 2cdH + FH^2 + FR^2, \end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{22} &= -48a^4 - 128a^3d + 16d^4 + 32d^2F^2 + F^4 + 12cdFH - 18d^2H^2 - 2F^2H^2 - 3H^4 \\
&\quad + 2a^2(4c^2 - 56d^2 + 9F^2 - 8H^2) + 2c^2(4d^2 - F^2 + H^2) + 32d^2R^2 + 2F^2R^2 - 2H^2R^2 \\
&\quad + R^4 - 4a(4c^2d - 7cFH - 2d(7F^2 - 3H^2 + 4R^2)), \\
\mathfrak{C}_{23} &= 12a^3F + 2c^2dF + a^2(44dF + 8cH) + dF(16d^2 + H^2) - cH(-2d^2 + F^2 + H^2) \\
&\quad + a(-2c^2F + 48d^2F + 14cdH - FH^2), \\
\mathfrak{C}_{24} &= -(2cdF + 4a^2H + a(-2cF + 8dH) + H(8d^2 + F^2 + H^2 + R^2)), \\
\mathfrak{C}_{33} &= 16a^4 + 128a^3d + 16d^4 - 32d^2F^2 + F^4 + 4cdFH - 14d^2H^2 + 2F^2H^2 + H^4 \\
&\quad + 2c^2(-4d^2 + F^2 + H^2) - 32d^2R^2 + 2F^2R^2 + 2H^2R^2 + R^4 - 2a^2(4c^2 - 104d^2 \\
&\quad + 7F^2 + 16H^2 + 16R^2) + 4a(4c^2d - cFH + 2d(16d^2 - 5F^2 - 5H^2 - 8R^2)), \\
\mathfrak{C}_{34} &= (-a + d)FH - c(F^2 + H^2), \\
\mathfrak{C}_{44} &= 16a^4 + 16d^4 + 16d^2F^2 + F^4 + 12cdFH + 14d^2H^2 + 2F^2H^2 + H^4 - 2c^2(4d^2 + F^2 + H^2) \\
&\quad + 2a^2(-4c^2 + 8d^2 + 7F^2 + 8H^2) + 4a(4c^2d - 3cFH + 2d(F^2 + H^2)) - 2F^2R^2 - 2H^2R^2 - 3R^4.
\end{aligned}$$

Hence a left-invariant metric determined by (3.6) is $\check{\rho}$ -Einstein if and only if the system of equations $\{\mathfrak{C}_{ij} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, H, F, R]$ the ideal generated by the polynomials $\mathfrak{C}_{ij}$. We compute a Gröbner basis $\mathcal{G} \subset \mathcal{I}$ with respect to the lexicographic order so that the polynomial $\mathbf{g}_0 = FH(F^2 + H^2 + R^2)^2(15F^2 + 15H^2 + 32R^2)$ belongs to $\mathcal{G}$. Hence $F = 0$ or $H = 0$, since $R > 0$.

Assume $F = 0$ and compute a Gröbner basis $\mathcal{G}_{01}$ of the ideal generated $\mathcal{G} \cup \{F\}$ with respect to the lexicographic order. Since the polynomial $\mathbf{g}_{01} = H^2(H^2 + R^2)(4H^2 + 9R^2)$ belongs to $\mathcal{G}_{01}$, one has that $H = 0$. On the other hand, assuming $H = 0$ and computing a Gröbner basis $\mathcal{G}_{02}$ of the ideal generated $\mathcal{G} \cup \{H\}$ with respect to the lexicographic order, one has that $\mathbf{g}_{02} = F^2(F^2 + R^2)(4F^2 + 9R^2) \in \mathcal{G}_{02}$, thus showing that $F = 0$.

We assume therefore that $F = H = 0$ in what follows. Then one has that

$$\mathfrak{C}_{12} = 4c(a - d)(2a^2 + 4ad + 2d^2 + R^2)$$

and the only possible real zeros are $c = 0$ and $a = d$.

If $c = 0$, then the only remaining equations are

$$\begin{aligned}
\mathfrak{C}_{11} &= 16a^4 - 112a^2d^2 - 128ad^3 - 48d^4 + 32a^2R^2 + 32adR^2 + R^4 \\
\mathfrak{C}_{22} &= -48a^4 - 128a^3d - 112a^2d^2 + 16d^4 + 32adR^2 + 32d^2R^2 + R^4 \\
\mathfrak{C}_{33} &= 16a^4 + 128a^3d + 16d^4 - 32d^2R^2 + R^4 + 16a^2(13d^2 - 2R^2) + 64a(2d^3 - dR^2) \\
\mathfrak{C}_{44} &= -3R^4 + 16(a^4 + a^2d^2 + d^4),
\end{aligned}$$

and the only possible solutions for the system are $d = \pm a$ and $2a = \pm R$. The cases corresponding to $d = a$ are Einstein so we assume $d = -a$ and $2a = \pm R$. A straightforward calculation now shows that Ricci operator $\text{Ric} = \frac{1}{2} \text{diag}[-1, -1, 1, -1]$ and hence the left-invariant metric is $\check{\rho}$ -Einstein. This shows (5).

Finally, assuming $a = d$, $c \neq 0$, one has that $\mathfrak{C}_{44} = 3(-R^4 + 16a^4)$. Hence $2a = \pm R$, and the corresponding left-invariant metrics are Einstein. $\square$

3.4. Left-invariant $\check{\rho}$ -Einstein metrics in $\mathbb{R}_{e_4} \ltimes \mathbb{R}^3$.

Lemma 6. $\mathbb{R}_{e_4} \ltimes \mathbb{R}^3$ admits a non-Einstein $\check{\rho}$ -Einstein left invariant metric if and only if it is homothetic to a Lie group determined by a solvable Lie algebra given by

$$\begin{aligned}
(3.7) \quad [e_4, e_1] &= e_1 + be_2 + ce_3, & [e_4, e_2] &= -be_1 + \alpha e_2 + he_3, \\
[e_4, e_3] &= -ce_1 - he_2 - \frac{\alpha}{\alpha+1}e_3,
\end{aligned}$$

where

$$b = -\frac{\alpha(1+\alpha+\alpha^2)^2}{(\alpha+1)^2(\alpha^3-1)}, \quad h = \frac{(1+\alpha+\alpha^2)^2}{\alpha(\alpha+2)}, \quad c = \frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}, \quad \alpha \in \mathbb{R}, \alpha \neq 0.$$

Proof. Let $\mathfrak{r}^3$ be the abelian algebra. The corresponding algebra of derivations is $\mathfrak{gl}(3, \mathbb{R})$. For any $D \in \mathfrak{gl}(3, \mathbb{R})$, decomposing it into its symmetric and skew-symmetric part, one has

$$\text{der}(\mathfrak{r}_3) = \left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix}; a, b, c, f, h, p \in \mathbb{R} \right\}.$$

The corresponding semi-direct product $\mathbb{R} \oplus \mathfrak{r}^3$ expresses in an orthonormal basis $\{e_1, \dots, e_4\}$ as

$$(3.8) \quad \begin{aligned} [e_4, e_1] &= \frac{1}{R}(ae_1 + be_2 + ce_3), & [e_4, e_2] &= \frac{1}{R}(-be_1 + fe_2 + he_3), \\ [e_4, e_3] &= \frac{1}{R}(-ce_1 - he_2 + pe_3), & R &> 0. \end{aligned}$$

Furthermore, considering the homothetic basis $\bar{e}_k = Re_k$, one may assume $R = 1$ in what follows. A straightforward calculation shows that $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ is determined by

$$\begin{aligned} 2R^4\check{\rho}_{011} &= \mathfrak{C}_{11}, & R^4\check{\rho}_{012} &= \mathfrak{C}_{12}, & R^4\check{\rho}_{013} &= \mathfrak{C}_{13}, & \check{\rho}_{014} &= 0, \\ 2R^4\check{\rho}_{022} &= \mathfrak{C}_{22}, & R^4\check{\rho}_{023} &= \mathfrak{C}_{23}, & \check{\rho}_{024} &= 0, & R^4\check{\rho}_{033} &= \mathfrak{C}_{33}, \\ \check{\rho}_{034} &= 0, & 2R^4\check{\rho}_{044} &= \mathfrak{C}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{C}_{ij}$ are given by

$$\begin{aligned} \mathfrak{C}_{11} &= -(-a^4 - b^2f^2 + f^4 + f^2h^2 + f^3p - 2fh^2p - c^2p^2 + 2f^2p^2 + h^2p^2 + fp^3 + p^4 \\ &\quad - 3a^3(f+p) - a^2(b^2 + c^2 + 3fp) + a(2b^2f + f^3 + 2c^2p + f^2p + fp^2 + p^3)), \\ \mathfrak{C}_{12} &= -a^3b + a(bf^2 + ch(f-p)) + chp(-f+p) - a^2b(f+p) + bf^2(f+p), \\ \mathfrak{C}_{13} &= -a^3c + bfh(-f+p) - a^2c(f+p) + cp^2(f+p) + a(bh(f-p) + cp^2), \\ \mathfrak{C}_{22} &= -(a^4 - b^2f^2 - f^4 - f^2h^2 - 3f^3p + 2fh^2p + c^2p^2 - h^2p^2 + fp^3 + p^4 \\ &\quad + a^3(f+p) + a(2b^2f - 3f^3 - 2c^2p - 3f^2p + fp^2 + p^3) + a^2(-b^2 + c^2 + p(f+2p))), \\ \mathfrak{C}_{23} &= a^2bc - f^3h - f^2hp + hp^3 - a(bc + h(f-p))(f+p) + fp(bc + hp), \\ \mathfrak{C}_{33} &= -(a^4 + b^2f^2 + f^4 - f^2h^2 + f^3p + 2fh^2p - c^2p^2 - h^2p^2 - 3fp^3 - p^4 \\ &\quad + a^3(f+p) + a(-2b^2f + f^3 + 2c^2p + f^2p - 3fp^2 - 3p^3) + a^2(b^2 - c^2 + f(2f+p))), \\ \mathfrak{C}_{44} &= -(-a^4 + b^2f^2 - f^4 + f^2h^2 + f^3p - 2fh^2p + c^2p^2 - 2f^2p^2 + h^2p^2 + fp^3 - p^4 \\ &\quad + a^3(f+p) + a^2(b^2 + c^2 - 2f^2 + fp - 2p^2) + a(-2b^2f + f^3 - 2c^2p + f^2p + fp^2 + p^3)). \end{aligned}$$

Further observe from (3.8) that the structure constants are symmetric (up to a change of basis) in the parameters b, c, h as well as in the parameters a, f, p . This clearly influence the polynomials $\mathfrak{C}_{ij}$. We are firstly considering the following four cases.

3.4.1. Assume $a = f$. Then $\mathfrak{C}_{12} = ch(f-p)^2$. One easily checks that if $f = p$ then the metric Einstein. Hence $f \neq p$ and either $c = 0$ or $h = 0$.

If $c = 0$, then $\mathfrak{C}_{23} = h(-2f^3 - f^2p + 2fp^2 + p^3)$. Now, if $h = 0$, then $\mathfrak{C}_{44} = (f-p)^2(2f^2+p^2)$, which contradicts $f \neq p$. If $h \neq 0$, then either $p = -2f$ or $p = -f$. Setting $p = -2f$ one gets $\mathfrak{C}_{11} = -9f^2(2f^2 + h^2)$. Hence $f = 0$ and thus $f = p = 0$ which is a contradiction. Finally, if $p = -f$, then one gets $\mathfrak{C}_{11} = -2f^2(f^2 + h^2)$, which also leads to a contradiction.

If $h = 0$ and $c \neq 0$, then $\mathfrak{C}_{13} = c(-2f^3 - f^2p + 2fp^2 + p^3)$ and hence $p = -2f$ or $p = -f$. Proceeding in a completely analogous way as in the previous case with the coefficient $\mathfrak{C}_{22}$ one gets that no $\check{\rho}$ -Einstein metrics may exist in this case.

Since the system of polynomial equations is symmetric up to a change of the basis $\{e_k\}$, the previous case also includes the situations $a = p$ and $f = p$. Hence we assume in what follows that $a \neq p$, $a \neq f$ and $p \neq f$.

3.4.2. Assume $a = -f$. Then $\mathfrak{C}_{12} = ch(-f^2 + p^2)$ and, since $a \neq p$, $a \neq f$ and $p \neq f$, one has $c = 0$ or $h = 0$.

If $c = 0$ then $\mathfrak{C}_{23} = hp(-f^2 + p^2)$ and thus $h = 0$ or $p = 0$. If $h = 0$, then we have the system of equations

$$\begin{aligned}\mathfrak{C}_{11} &= \mathfrak{C}_{22} = 4b^2f^2 - 2f^4 - f^2p^2 - p^4 \\ \mathfrak{C}_{33} &= -4b^2f^2 - 2f^4 - 3f^2p^2 + p^4 \\ \mathfrak{C}_{44} &= -4b^2f^2 + 6f^4 + 5f^2p^2 + p^4,\end{aligned}$$

which has no real solutions. If $p = 0$, $h \neq 0$ then $\mathfrak{C}_{13} = -2bf^2h$. Hence $b = 0$ and then $\mathfrak{C}_{11} = -f^2(2f^2 + h^2)$, which has no zeroes. This shows that c cannot be zero.

If $h = 0$, $c \neq 0$ then $\mathfrak{C}_{13} = cp(-f^2 + p^2)$ and the only possibility is $p = 0$, but then $\mathfrak{C}_{23} = 2bcf^2$, which implies that $b = 0$ and thus $\mathfrak{C}_{22} = -f^2(c^2 + 2f^2)$, which has no zeroes.

Therefore, we have seen that $a \neq -f$. Hence, by considering appropriate changes on the basis $\{e_k\}$ one has that $a^2 \neq f^2$, $a^2 \neq p^2$, $f^2 \neq p^2$.

3.4.3. Assume $c = 0$. Then $\mathfrak{C}_{12} = -b(a^2 - f^2)(a + f + p)$, and thus either $b = 0$ or $a + f + p = 0$.

If $a + f + p = 0$ then $\mathfrak{C}_{13} = bh(-2f^2 + fp + p^2)$, and hence either $b = 0$, $h = 0$ or $p = -2f$. Note that the later condition is not possible since $a + f + p = 0$ and $p = -2f$ would give $a = f$, which is a contradiction. If $b = 0$, then we have

$$\begin{aligned}\mathfrak{C}_{11} &= 2(-f^4 - 2f^3p + fp(h^2 - 2p^2)) - p^2(h^2 + 2p^2) - f^2(h^2 + 6p^2) \\ \mathfrak{C}_{22} &= \mathfrak{C}_{33} = -2f^4 - 4f^3p + f^2(h^2 - 6p^2) + p^2(h^2 - 2p^2) - 2fp(h^2 + 2p^2) \\ \mathfrak{C}_{44} &= -h^2(f-p)^2 + 6(f^2 + fp + p^2)^2,\end{aligned}$$

which has no real solution. Assuming $h = 0$ and $b \neq 0$, one has

$$\begin{aligned}\mathfrak{C}_{11} &= \mathfrak{C}_{22} = b^2(2f+p)^2 - 2(f^2 + fp + p^2)^2 \\ \mathfrak{C}_{33} &= -b^2(2f+p)^2 - 2(f^2 + fp + p^2)^2 \\ \mathfrak{C}_{44} &= -h^2(f-p)^2 + 6(f^2 + fp + p^2)^2,\end{aligned}$$

which again has no real solution.

Set now $b = 0$ and assume $a + f + p \neq 0$. Then $\mathfrak{C}_{23} = -h(a + f + p)(f^2 - p^2)$ shows that $h = 0$. Now the system of polynomial equations becomes

$$\begin{aligned}\mathfrak{C}_{11} &= a^4 - f^4 + 3a^2fp - f^3p - 2f^2p^2 - fp^3 - p^4 \\ &\quad + 3a^3(f + p) - a(f^3 + f^2p + fp^2 + p^3) \\ \mathfrak{C}_{22} &= -a^4 + f^4 + 3f^3p - fp^3 - p^4 - a^3(f + p) \\ &\quad - a^2p(f + 2p) - a(-3f^3 - 3f^2p + fp^2 + p^3) \\ \mathfrak{C}_{33} &= -a^4 - f^4 - f^3p + 3fp^3 + p^4 - a^3(f + p) \\ &\quad - a^2f(2f + p) - a(f^3 + f^2p - 3fp^2 - 3p^3) \\ \mathfrak{C}_{44} &= -2a^2(a + f + p)^2 - f^2(a + f + p)^2 \\ &\quad - p^2(a + f + p)^2 + 3(a^2 + f^2 + p^2)^2,\end{aligned}$$

and again it has no real solution.

Hence we may assume $c \neq 0$ and, up to a change of basis, one has that $c \neq 0$, $b \neq 0$ and $h \neq 0$.

3.4.4. Assume $a + f + p = 0$. Then $\mathfrak{C}_{12} = -ch(f^2 + fp - 2p^2)$ and thus $p = f$ or $p = -\frac{1}{2}f$, which contradicts $a^2 \neq f^2$ and $p^2 \neq a^2$.

We now return to the original system $\{\mathfrak{C}_{ij} = 0\}$. Based on the analysis of the previous cases, the polynomial

$$\mathfrak{C}_{12} = -a^3b + a(bf^2 + ch(f - p)) + chp(-f + p) - a^2b(f + p) + bf^2(f + p),$$

gives $b = \frac{ch(a - p)(f - p)}{(a^2 - f^2)(a + f + p)}$. Substituting b in $\mathfrak{C}_{13}$, it becomes

$$\begin{aligned}\frac{(a+f)(a+f+p)}{c(p-a)}\mathfrak{C}_{13} &= a^4 + f^3p - h^2p^2 + 3a^3(f + p) - f^2(h^2 - 2p^2) \\ &\quad + a^2(3f^2 + 7fp + 3p^2) + f(2h^2p + p^3) + a(f^3 + 5f^2p + 5fp^2 + p^3).\end{aligned}$$

Hence $h^2 = \frac{(a + f)(a + p)(a + f + p)^2}{(f - p)^2}$. Replacing again in the equations, one gets

$$\begin{aligned}\frac{(a+f)(a+f+p)}{h(p-f)}\mathfrak{C}_{23} &= f^4 + 3f^3p - c^2p^2 + 3f^2p^2 + fp^3 + a^3(f + p) \\ &\quad + a^2(-c^2 + 3f^2 + 5fp + 2p^2) + a(3f^3 + 2c^2p + 7f^2p + 5fp^2 + p^3),\end{aligned}$$

and thus $c^2 = \frac{(a + f)(f + p)(a + f + p)^2}{(a - p)^2}$. Now, the only remaining equations are

$\mathfrak{C}_{11} = \mathfrak{C}_{22} = \mathfrak{C}_{33} = \mathfrak{C}_{44}$, where

$$\mathfrak{C}_{ii} = a^3(f + p) + a(f + p)^3 + fp(f^2 + fp + p^2) + a^2(f^2 + 3fp + p^2).$$

The only real solutions of $\mathfrak{C}_{ii} = 0$ are $a = f = 0$ (which gives an Einstein metric)

and $p = -\frac{af}{a + f}$.

Next observe that setting $\hat{e}_1 = e_2$, $\hat{e}_2 = e_1$, $\hat{e}_3 = e_3$, $\hat{e}_4 = e_4$ one gets an isometry interchanging a and f . Furthermore since $p \neq a$, $p \neq f$ the constants a, f satisfy $af \neq 0$. Hence considering the homothetic basis $\tilde{e}_k = \frac{1}{a}e_k$ one may assume that $a = 1$. Finally setting $f = \alpha$ one has that $p = -\frac{\alpha}{\alpha+1}$ and the remaining structure constants are given by

$$h = \varepsilon_h \frac{(1 + \alpha + \alpha^2)^2}{\alpha(\alpha + 2)}, \quad c = \varepsilon_c \frac{\alpha(1 + \alpha + \alpha^2)}{2\alpha + 1}, \quad b = -\varepsilon_h \varepsilon_c \frac{\alpha(1 + \alpha + \alpha^2)^2}{(\alpha + 1)^2(\alpha^3 - 1)},$$

where $\varepsilon_h^2 = \varepsilon_c^2 = 1$. Finally since $e_1^* = -e_1$, $e_2^* = -e_2$ (resp., $e_1^* = -e_1$, $e_3^* = -e_3$ and $e_2^* = -e_2$, $e_3^* = -e_3$) determines an isometry interchanging the signs of c and h (resp., the signs of b and h and the signs of b and c) we may assume $\varepsilon_h = \varepsilon_c = 1$. We recall that the values $\alpha = 0$, $\alpha = -2$, $\alpha = -\frac{1}{2}$ and $\alpha = \pm 1$ are not possible since $a^2 \neq f^2$, $a^2 \neq p^2$ and $f^2 \neq p^2$. Furthermore, the isometry interchanging a and f induces an isometry interchanging α with $\frac{1}{\alpha}$.

Finally observe that the Ricci operator takes the form

$$\text{Ric} = \frac{1 + \alpha + \alpha^2}{\alpha + 1} \begin{pmatrix} -1 & \frac{\alpha}{\alpha+1} & \alpha & 0 \\ \frac{\alpha}{\alpha+1} & -\alpha & 1 & 0 \\ \alpha & 1 & \frac{\alpha}{\alpha+1} & 0 \\ 0 & 0 & 0 & -\frac{1+\alpha+\alpha^2}{\alpha+1} \end{pmatrix}$$

which shows that the left-invariant metrics (3.7) are $\check{\rho}$ -Einstein but not Einstein. $\square$

4. LEFT-INVARIANT $R[\rho]$ -EINSTEIN METRICS

We show that, in contrast with the weakly-Einstein and the $\check{\rho}$ -Einstein situations, a four-dimensional homogeneous $R[\rho]$ -Einstein metric is necessarily Einstein or the symmetric product given in Lemma 2. The result follows after a case by case analysis of the left-invariant metrics on four-dimensional Lie groups.

4.1. Left-invariant $R[\rho]$ -Einstein metrics in $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ and $SU(\mathbb{R}) \times \mathbb{R}$.

Lemma 7. *$\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ and $SU(\mathbb{R}) \times \mathbb{R}$ do not admit $R[\rho]$ -Einstein left invariant metrics.*

Proof. Let $R[\rho]_0$ be the trace-free tensor $R[\rho]_0 = R[\rho] - \frac{1}{4}\|\rho\|^2 g$. A straightforward calculation shows that the components of $R[\rho]_0$ for any left-invariant metric (3.3) are given by

$$\begin{aligned} 16R^4R[\rho]_{011} &= \mathfrak{R}_{11}, & 8R^4R[\rho]_{012} &= \mathfrak{R}_{12}, & 8R^4R[\rho]_{013} &= \mathfrak{R}_{13}, & 8R^3R[\rho]_{014} &= \mathfrak{R}_{14}, \\ 16R^4R[\rho]_{022} &= \mathfrak{R}_{22}, & 8R^4R[\rho]_{023} &= \mathfrak{R}_{23}, & 8R^3R[\rho]_{024} &= \mathfrak{R}_{24}, & 16R^4R[\rho]_{033} &= \mathfrak{R}_{33}, \\ 8R^3R[\rho]_{034} &= \mathfrak{R}_{34}, & 16R^4R[\rho]_{044} &= \mathfrak{R}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{R}_{ij}$ are

$$\begin{aligned} \mathfrak{R}_{11} &= -7R^4\lambda_1^4 + 8R^4\lambda_1^3\lambda_2 + 10R^4\lambda_1^2\lambda_2^2 - 4k_1^2R^2\lambda_1\lambda_2^3 - 16R^4\lambda_1\lambda_2^3 - 3k_1^4\lambda_2^4 + 2k_1^2R^2\lambda_2^4 + 5R^4\lambda_2^4 \\ &\quad - k_3^4(\lambda_1 - \lambda_2)^2(7\lambda_1^2 + 6\lambda_1\lambda_2 - 5\lambda_2^2) + 8R^4\lambda_1^3\lambda_3 - 28R^4\lambda_1^2\lambda_2\lambda_3 + 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3 + 16R^4\lambda_1\lambda_2^2\lambda_3 \\ &\quad + 4k_1^4\lambda_2^3\lambda_3 + 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 + 10R^4\lambda_1^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_2\lambda_3^2 + 16R^4\lambda_1\lambda_2\lambda_3^2 - 2k_1^4\lambda_2^2\lambda_3^2 \\ &\quad - 20k_1^2R^2\lambda_2^2\lambda_3^2 - 18R^4\lambda_2^2\lambda_3^2 - 4k_1^2R^2\lambda_1\lambda_3^3 - 16R^4\lambda_1\lambda_3^3 + 4k_1^4\lambda_2\lambda_3^3 + 8k_1^2R^2\lambda_2\lambda_3^3 \\ &\quad + 4R^4\lambda_2\lambda_3^3 - 3k_1^4\lambda_3^4 + 2k_1^2R^2\lambda_3^4 + 5R^4\lambda_3^4 - k_2^4(\lambda_1 - \lambda_3)^2(7\lambda_1^2 + 6\lambda_1\lambda_3 - 5\lambda_3^2) \\ &\quad + 2k_3^2(-R^2(\lambda_1 - \lambda_2)(7\lambda_1^3 - \lambda_1^2(\lambda_2 + 4\lambda_3) + \lambda_2(5\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2)) \\ &\quad + \lambda_1(-11\lambda_2^2 + 10\lambda_2\lambda_3 - 2\lambda_3^2)) + k_1^2(\lambda_1^2(4\lambda_2 - 5\lambda_3)\lambda_3 + 2\lambda_1\lambda_2(-\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2) \\ &\quad + \lambda_2^2(\lambda_2^2 - 4\lambda_2\lambda_3 + 2\lambda_3^2))) + 2k_2^2(k_1^2(\lambda_1^2\lambda_2(-5\lambda_2 + 4\lambda_3) + 2\lambda_1\lambda_3(\lambda_2^2 + \lambda_2\lambda_3 - \lambda_3^2) \\ &\quad + \lambda_3^2(2\lambda_2^2 - 4\lambda_2\lambda_3 + \lambda_3^2)) + k_3^2(-7\lambda_1^4 + 5\lambda_2^2\lambda_3^2 + 4\lambda_1^3(\lambda_2 + \lambda_3) - 8\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3) \\ &\quad + 2\lambda_1^2(\lambda_2^2 + 3\lambda_2\lambda_3 + \lambda_3^2)) - R^2(\lambda_1 - \lambda_3)(7\lambda_1^3 - \lambda_1^2(4\lambda_2 + \lambda_3) + \lambda_1(-2\lambda_2^2 + 10\lambda_2\lambda_3 - 11\lambda_3^2) \\ &\quad + \lambda_3(-4\lambda_2^2 + 2\lambda_2\lambda_3 + 5\lambda_3^2))), \\ \mathfrak{R}_{12} &= k_1k_2(R^2\lambda_1^3\lambda_2 - 2R^2\lambda_1^2\lambda_2^2 + 2k_1^2\lambda_1\lambda_2^3 + R^2\lambda_1\lambda_2^3 + 5R^2\lambda_1^2\lambda_2\lambda_3 - 2k_1^2\lambda_1\lambda_2^2\lambda_3 + 5R^2\lambda_1\lambda_2^2\lambda_3 \\ &\quad - 4R^2\lambda_1^2\lambda_3^2 - 2k_1^2\lambda_1\lambda_2\lambda_3^2 - 14R^2\lambda_1\lambda_2\lambda_3^2 - 4k_1^2\lambda_2^2\lambda_3^2 - 4R^2\lambda_2^2\lambda_3^2 + 2k_1^2\lambda_1\lambda_3^3 + 8R^2\lambda_1\lambda_3^3 \\ &\quad + 8k_1^2\lambda_2\lambda_3^3 + 8R^2\lambda_2\lambda_3^3 - 4k_1^2\lambda_3^4 - 4R^2\lambda_3^4 + 2k_2^2(\lambda_1 - \lambda_3)^2(\lambda_1\lambda_2 + (\lambda_2 - 2\lambda_3)\lambda_3)) \end{aligned}$$

$$\begin{aligned}
& + k_3^2(\lambda_1 - \lambda_2)^2((\lambda_2 - 3\lambda_3)\lambda_3 + \lambda_1(\lambda_2 + \lambda_3))), \\
\mathfrak{R}_{13} = & k_1k_3(-4R^2\lambda_1^2\lambda_2^2 + 2k_1^2\lambda_1\lambda_2^3 + 8R^2\lambda_1\lambda_2^3 - 4k_1^2\lambda_2^4 - 4R^2\lambda_2^4 + R^2\lambda_1^3\lambda_3 + 5R^2\lambda_1^2\lambda_2\lambda_3 \\
& - 2k_1^2\lambda_1\lambda_2^2\lambda_3 - 14R^2\lambda_1\lambda_2^2\lambda_3 + 8k_1^2\lambda_2^3\lambda_3 + 8R^2\lambda_2^3\lambda_3 - 2R^2\lambda_1^2\lambda_3^2 - 2k_1^2\lambda_1\lambda_2\lambda_3^2 + 5R^2\lambda_1\lambda_2\lambda_3^2 \\
& - 4k_1^2\lambda_2^2\lambda_3^2 - 4R^2\lambda_2^2\lambda_3^2 + 2k_1^2\lambda_1\lambda_3^3 + R^2\lambda_1\lambda_3^3 + 2k_3^2(\lambda_1 - \lambda_2)^2(-2\lambda_2^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \\
& + k_2^2(\lambda_1 - \lambda_3)^2(\lambda_2(-3\lambda_2 + \lambda_3) + \lambda_1(\lambda_2 + \lambda_3))), \\
\mathfrak{R}_{14} = & k_1(k_3^2(\lambda_1 - \lambda_2)(-4\lambda_1\lambda_2^2 + 4\lambda_2^3 + \lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 + 6\lambda_2^2\lambda_3 - 3\lambda_1\lambda_3^2 - 6\lambda_2\lambda_3^2) \\
& + k_2^2(\lambda_1 - \lambda_3)(\lambda_1^2\lambda_2 - 6\lambda_2^2\lambda_3 + 6\lambda_2\lambda_3^2 + 4\lambda_3^3 + \lambda_1(-3\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2)) \\
& - 2(\lambda_2 - \lambda_3)^2(2R^2(-\lambda_1 + \lambda_2 + \lambda_3)^2 - k_1^2(\lambda_2 + \lambda_3)(\lambda_1 - 2(\lambda_2 + \lambda_3)))), \\
\mathfrak{R}_{22} = & 5R^4\lambda_1^4 - 16R^4\lambda_1^3\lambda_2 + 4k_1^2R^2\lambda_1^2\lambda_2^2 + 10R^4\lambda_1^2\lambda_2^2 + 8k_1^2R^2\lambda_1\lambda_2^3 + 8R^4\lambda_1\lambda_2^3 - 7k_1^4\lambda_2^4 \\
& - 14k_1^2R^2\lambda_2^4 - 7R^4\lambda_2^4 + k_3^4(\lambda_1 - \lambda_2)^2(5\lambda_1^2 - 6\lambda_1\lambda_2 - 7\lambda_2^2) + 4R^4\lambda_1^3\lambda_3 + 4k_1^2R^2\lambda_1^2\lambda_2\lambda_3 \\
& + 16R^4\lambda_1^2\lambda_2\lambda_3 - 28k_1^2R^2\lambda_1\lambda_2^2\lambda_3 - 28R^4\lambda_1\lambda_2^2\lambda_3 + 8k_1^4\lambda_2^3\lambda_3 + 16k_1^2R^2\lambda_2^3\lambda_3 + 8R^4\lambda_2^3\lambda_3 \\
& - 8k_1^2R^2\lambda_1^2\lambda_3^2 - 18R^4\lambda_1^2\lambda_3^2 + 16k_1^2R^2\lambda_1\lambda_2\lambda_3^2 + 16R^4\lambda_1\lambda_2\lambda_3^2 + 10k_1^4\lambda_2^2\lambda_3^2 + 20k_1^2R^2\lambda_2^2\lambda_3^2 \\
& + 10R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3 - 16k_1^4\lambda_2\lambda_3^3 - 32k_1^2R^2\lambda_2\lambda_3^3 - 16R^4\lambda_2\lambda_3^3 + 5k_1^4\lambda_3^4 \\
& + 10k_1^2R^2\lambda_3^4 + 5R^4\lambda_3^4 - k_2^4(\lambda_1 - \lambda_3)^2(3\lambda_1^2 + 2\lambda_1\lambda_3 + 3\lambda_3^2) \\
& + 2k_2^2(R^2(\lambda_1 - \lambda_3)^2(\lambda_1^2 - 2\lambda_1\lambda_2 + 6\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2) + k_3^2(\lambda_1^4 - 5\lambda_2^2\lambda_3^2 + 2\lambda_1^2\lambda_3(\lambda_2 + \lambda_3) \\
& + 2\lambda_1\lambda_2\lambda_3(2\lambda_2 + \lambda_3) - 2\lambda_1^3(\lambda_2 + 2\lambda_3)) + k_1^2(\lambda_3^3(-2\lambda_2 + \lambda_3) + 2\lambda_1\lambda_3(2\lambda_2^2 + \lambda_2\lambda_3 - 2\lambda_3^2) \\
& + \lambda_1^2(-5\lambda_2^2 + 2\lambda_2\lambda_3 + 2\lambda_3^2))) + 2k_3^2(R^2(\lambda_1 - \lambda_2)(5\lambda_1^3 + \lambda_1^2(-11\lambda_2 + 2\lambda_3) \\
& + \lambda_2(7\lambda_2^2 - 4\lambda_2\lambda_3 - 2\lambda_3^2) - \lambda_1(\lambda_2^2 - 10\lambda_2\lambda_3 + 4\lambda_3^2)) + k_1^2(2\lambda_1\lambda_2(2\lambda_2^2 + 3\lambda_2\lambda_3 - 4\lambda_3^2) \\
& + \lambda_2^2(-7\lambda_2^2 + 4\lambda_2\lambda_3 + 2\lambda_3^2) + \lambda_1^2(2\lambda_2^2 - 8\lambda_2\lambda_3 + 5\lambda_3^2))), \\
\mathfrak{R}_{23} = & k_2k_3(-4R^2\lambda_1^4 + 8R^2\lambda_1^3\lambda_2 - 3k_1^2\lambda_1^2\lambda_2^2 - 4R^2\lambda_1^2\lambda_2^2 + k_1^2\lambda_1\lambda_2^3 + 8R^2\lambda_1^3\lambda_3 + 6k_1^2\lambda_1^2\lambda_2\lambda_3 \\
& - 14R^2\lambda_1^2\lambda_2\lambda_3 - k_1^2\lambda_1\lambda_2^2\lambda_3 + 5R^2\lambda_1\lambda_2^2\lambda_3 + k_1^2\lambda_2^3\lambda_3 + R^2\lambda_2^3\lambda_3 - 3k_1^2\lambda_1^2\lambda_3^2 - 4R^2\lambda_1^2\lambda_3^2 \\
& - k_1^2\lambda_1\lambda_2\lambda_3^2 + 5R^2\lambda_1\lambda_2\lambda_3^2 - 2k_1^2\lambda_2^2\lambda_3^2 - 2R^2\lambda_2^2\lambda_3^2 + k_1^2\lambda_1\lambda_3^3 + k_1^2\lambda_2\lambda_3^3 + R^2\lambda_2\lambda_3^3 \\
& - 2k_2^2(\lambda_1 - \lambda_3)^2(2\lambda_1^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3) - 2k_3^2(\lambda_1 - \lambda_2)^2(2\lambda_1^2 - \lambda_1\lambda_3 - \lambda_2\lambda_3)), \\
\mathfrak{R}_{24} = & -k_2(4R^2\lambda_1^4 - 8R^2\lambda_1^3\lambda_2 + 3k_1^2\lambda_1^2\lambda_2^2 + 4R^2\lambda_1^2\lambda_2^2 - k_1^2\lambda_1\lambda_2^3 + 3k_1^2\lambda_1^2\lambda_2\lambda_3 + 8R^2\lambda_1^2\lambda_2\lambda_3 \\
& - k_1^2\lambda_1\lambda_2^2\lambda_3 - 8R^2\lambda_1\lambda_2^2\lambda_3 - 6k_1^2\lambda_1^2\lambda_3^2 - 8R^2\lambda_1^2\lambda_3^2 - 4k_1^2\lambda_1\lambda_2\lambda_3^2 + 8R^2\lambda_1\lambda_2\lambda_3^2 + 4k_1^2\lambda_2^2\lambda_3^2 \\
& + 4R^2\lambda_2^2\lambda_3^2 + 6k_1^2\lambda_1\lambda_3^3 - 8k_1^2\lambda_2\lambda_3^3 - 8R^2\lambda_2\lambda_3^3 + 4k_1^4\lambda_3^4 + 4R^2\lambda_3^4 \\
& + 2k_2^2(\lambda_1 - \lambda_3)^2(\lambda_1 + \lambda_3)(2\lambda_1 - \lambda_2 + 2\lambda_3) + k_3^2(\lambda_1 - \lambda_2)(4\lambda_1^3 + 2\lambda_1(\lambda_2 - 3\lambda_3)\lambda_3 \\
& + \lambda_2(\lambda_2 - 3\lambda_3)\lambda_3 + \lambda_1^2(-4\lambda_2 + 6\lambda_3))), \\
\mathfrak{R}_{33} = & 5R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 - 8k_1^2R^2\lambda_1^2\lambda_2^2 - 18R^4\lambda_1^2\lambda_2^2 + 4k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 + 5k_1^4\lambda_2^4 \\
& + 10k_1^2R^2\lambda_2^4 + 5R^4\lambda_2^4 - k_3^4(\lambda_1 - \lambda_2)^2(3\lambda_1^2 + 2\lambda_1\lambda_2 + 3\lambda_2^2) - 16R^4\lambda_1^3\lambda_3 + 4k_1^2R^2\lambda_1^2\lambda_2\lambda_3 \\
& + 16R^4\lambda_1^2\lambda_2\lambda_3 + 16k_1^2R^2\lambda_1\lambda_2^2\lambda_3 + 16R^4\lambda_1\lambda_2^2\lambda_3 - 16k_1^4\lambda_2^3\lambda_3 - 32k_1^2R^2\lambda_2^3\lambda_3 - 16R^4\lambda_2^3\lambda_3 \\
& + 4k_1^2R^2\lambda_1^2\lambda_3^2 + 10R^4\lambda_1^2\lambda_3^2 - 28k_1^2R^2\lambda_1\lambda_2\lambda_3^2 - 28R^4\lambda_1\lambda_2\lambda_3^2 + 10k_1^4\lambda_2^2\lambda_3^2 + 20k_1^2R^2\lambda_2^2\lambda_3^2 \\
& + 10R^4\lambda_2^2\lambda_3^2 + 8k_1^2R^2\lambda_1\lambda_3^3 + 8R^4\lambda_1\lambda_3^3 + 8k_1^4\lambda_2\lambda_3^3 + 16k_1^2R^2\lambda_2\lambda_3^3 + 8R^4\lambda_2\lambda_3^3 - 7k_1^4\lambda_3^4 \\
& - 14k_1^2R^2\lambda_3^4 - 7R^4\lambda_3^4 + k_2^4(\lambda_1 - \lambda_3)^2(5\lambda_1^2 - 6\lambda_1\lambda_3 - 7\lambda_3^2) \\
& + 2k_3^2(R^2(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3) + k_1^2(\lambda_3^3(\lambda_2 - 2\lambda_3) \\
& + \lambda_1^2(2\lambda_2^2 + 2\lambda_2\lambda_3 - 5\lambda_3^2) + 2\lambda_1\lambda_2(-2\lambda_2^2 + \lambda_2\lambda_3 + 2\lambda_3^2))) + 2k_2^2(k_3^2(\lambda_1^4 - 5\lambda_2^2\lambda_3^2 \\
& + 2\lambda_1^2\lambda_2(\lambda_2 + \lambda_3) - 2\lambda_1^3(2\lambda_2 + \lambda_3) + 2\lambda_1\lambda_2\lambda_3(\lambda_2 + 2\lambda_3)) \\
& + R^2(\lambda_1 - \lambda_3)(5\lambda_1^3 + \lambda_1^2(2\lambda_2 - 11\lambda_3) - \lambda_1(4\lambda_2^2 - 10\lambda_2\lambda_3 + \lambda_3^2) \\
& + \lambda_3(-2\lambda_2^2 - 4\lambda_2\lambda_3 + 7\lambda_3^2)) + k_1^2(\lambda_3^2(2\lambda_2^2 + 4\lambda_2\lambda_3 - 7\lambda_3^2) + \lambda_1^2(5\lambda_2^2 - 8\lambda_2\lambda_3 + 2\lambda_3^2) \\
& + \lambda_1(-8\lambda_2^2\lambda_3 + 6\lambda_2\lambda_3^2 + 4\lambda_3^3))), \\
\mathfrak{R}_{34} = & -k_3(4R^2\lambda_1^4 - 6k_1^2\lambda_1^2\lambda_2^2 - 8R^2\lambda_1^2\lambda_2^2 + 6k_1^2\lambda_1\lambda_2^3 + 4k_1^2\lambda_2^4 + 4R^2\lambda_2^4 \\
& + 2k_3^2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)(2\lambda_1 + 2\lambda_2 - \lambda_3) - 8R^2\lambda_1^3\lambda_3 + 3k_1^2\lambda_1^2\lambda_2\lambda_3 + 8R^2\lambda_1^2\lambda_2\lambda_3 - 4k_1^2\lambda_1\lambda_2^2\lambda_3 \\
& + 8R^2\lambda_1\lambda_2^2\lambda_3 - 8k_1^2\lambda_2^3\lambda_3 - 8R^2\lambda_2^3\lambda_3 + 3k_1^2\lambda_1^2\lambda_3^2 + 4R^2\lambda_1^2\lambda_3^2 - k_1^2\lambda_1\lambda_2\lambda_3^2 - 8R^2\lambda_1\lambda_2\lambda_3^2 \\
& + 4k_1^2\lambda_2^2\lambda_3^2 + 4R^2\lambda_2^2\lambda_3^2 - k_1^2\lambda_1\lambda_3^3 + k_2^2(\lambda_1 - \lambda_3)(4\lambda_1^3 + \lambda_1^2(6\lambda_2 - 4\lambda_3) \\
& + 2\lambda_1\lambda_2(-3\lambda_2 + \lambda_3) + \lambda_2\lambda_3(-3\lambda_2 + \lambda_3))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R}_{44} = & -3R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 + 4k_1^2R^2\lambda_1^2\lambda_2^2 - 2R^4\lambda_1^2\lambda_2^2 - 8k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 + 5k_1^4\lambda_2^4 \\
& + 2k_1^2R^2\lambda_2^4 - 3R^4\lambda_2^4 + k_3^4(\lambda_1 - \lambda_2)^2(5\lambda_1^2 + 14\lambda_1\lambda_2 + 5\lambda_2^2) + 4R^4\lambda_1^3\lambda_3 - 8k_1^2R^2\lambda_1^2\lambda_2\lambda_3 \\
& - 4R^4\lambda_1^2\lambda_2\lambda_3 + 8k_1^2R^2\lambda_1\lambda_2^2\lambda_3 - 4R^4\lambda_1\lambda_2^2\lambda_3 + 4k_1^4\lambda_2^3\lambda_3 + 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 + 4k_1^2R^2\lambda_1^2\lambda_3^2 \\
& - 2R^4\lambda_1^2\lambda_3^2 + 8k_1^2R^2\lambda_1\lambda_2\lambda_3^2 - 4R^4\lambda_1\lambda_2\lambda_3^2 - 18k_1^4\lambda_2^2\lambda_3^2 - 20k_1^2R^2\lambda_2^2\lambda_3^2 - 2R^4\lambda_2^2\lambda_3^2 - 8k_1^2R^2\lambda_1\lambda_3^3 \\
& + 4R^4\lambda_1\lambda_3^3 + 4k_1^4\lambda_2\lambda_3^3 + 8k_1^2R^2\lambda_2\lambda_3^3 + 4R^4\lambda_2\lambda_3^3 + 5k_1^4\lambda_3^4 + 2k_1^2R^2\lambda_3^4 - 3R^4\lambda_3^4 \\
& + k_2^4(\lambda_1 - \lambda_3)^2(5\lambda_1^2 + 14\lambda_1\lambda_3 + 5\lambda_3^2) + 2k_3^2(R^2(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 4\lambda_1\lambda_3 - 4\lambda_2\lambda_3 \\
& + 2\lambda_3^2) + k_1^2(\lambda_2^2(5\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2) + 2\lambda_1\lambda_2(\lambda_2^2 - 5\lambda_2\lambda_3 + \lambda_3^2) \\
& + \lambda_1^2(-4\lambda_2^2 + 2\lambda_2\lambda_3 + 5\lambda_3^2))) + 2k_2^2(R^2(\lambda_1 - \lambda_3)^2(\lambda_1^2 - 4\lambda_1\lambda_2 + 2\lambda_2^2 + 6\lambda_1\lambda_3 \\
& - 4\lambda_2\lambda_3 + \lambda_3^2) + k_3^2(5\lambda_1^4 + 5\lambda_2^2\lambda_3^2 + 2\lambda_1^3(\lambda_2 + \lambda_3) + 2\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3) \\
& - 2\lambda_1^2(2\lambda_2^2 + 5\lambda_2\lambda_3 + 2\lambda_3^2)) + k_1^2(\lambda_1^2(5\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2) + 2\lambda_1\lambda_3(\lambda_2^2 - 5\lambda_2\lambda_3 + \lambda_3^2) \\
& + \lambda_3^2(-4\lambda_2^2 + 2\lambda_2\lambda_3 + 5\lambda_3^2))).
\end{aligned}$$

We proceed as in the proof of Lemma 3. Assuming $\lambda_1 = \lambda_2 = \lambda_3$, one has that $R[\rho]_0 = \frac{1}{16} \text{diag}(\lambda_1^4, \lambda_1^4, \lambda_1^4, -3\lambda_1^4)$, thus showing that the metric is not $R[\rho]$ -Einstein since $\lambda_1\lambda_2\lambda_3 \neq 0$. Now, we consider the terms k_i at (3.3). Up to a permutation of the basis $\{e_1, e_2, e_3\}$ one may assume that one of the following holds:

$$(i) \ k_1k_2k_3 \neq 0, \quad (ii) \ k_1 = 0, k_2k_3 \neq 0, \quad (iii) \ k_1 = k_2 = 0, k_3 \neq 0.$$

We analyze the different possibilities separately.

4.1.1. (i) $k_1k_2k_3 \neq 0$. Let $\mathcal{I} \subset \mathbb{R}[k_1, k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ be the ideal generated by the polynomials $\mathfrak{R}_{ij}$ (after simplification due to $k_1k_2k_3 \neq 0$) and compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}$ with respect to the lexicographical order. Since the polynomial $\mathbf{g}_1 = R^4\lambda_1\lambda_2^2\lambda_3^2(\lambda_2^2 - \lambda_3^2)$ belongs to $\mathcal{G}_1$ one has that $\lambda_2^2 - \lambda_3^2 = 0$. Computing a Gröbner basis $\mathcal{G}_{11}$ of the ideal generated by $\mathcal{G}_1 \cup \{\lambda_2^2 - \lambda_3^2\}$, one has that $R^4\lambda_1^2\lambda_3^3$ belongs to $\mathcal{G}_{11}$, which is a contradiction.

4.1.2. (ii) $k_1 = 0, k_2k_3 \neq 0$. We firstly simplify the polynomials $\mathfrak{R}_{23}, \mathfrak{R}_{24}$ and $\mathfrak{R}_{34}$ by using that $k_2k_3 \neq 0$ and then compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{I} \subset \mathbb{R}[k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ generated by the polynomials $\mathfrak{R}_{ij}$ (after simplification). Since $\mathbf{g}_2 = R^6\lambda_2^2\lambda_3^3(\lambda_2 - \lambda_3) \in \mathcal{G}_2$ one has that $\lambda_2 = \lambda_3$. Computing a Gröbner basis $\mathcal{G}_{21}$ of the ideal generated by $\mathcal{G}_2 \cup \{\lambda_2 - \lambda_3\}$, one has that $R^4\lambda_1\lambda_3^4$ belongs to $\mathcal{G}_{21}$, which is a contradiction.

4.1.3. (iii) $k_1 = k_2 = 0, k_3 \neq 0$. Let $\mathcal{I} \subset \mathbb{R}[k_3, \lambda_1, \lambda_2, \lambda_3, R]$ be the ideal generated by the polynomials $\mathfrak{R}_{ij}$ (after simplifying $\mathfrak{R}_{34}$ due to $k_3 \neq 0$) and compute a Gröbner basis $\mathcal{G}_3$ of $\mathcal{I}$ with respect to the lexicographical order. Since the polynomial $\mathbf{g}_3 = R^6\lambda_2^2\lambda_3^3(\lambda_2^2 - \lambda_3^2)^2$ belongs to $\mathcal{G}_3$ one has that $\lambda_2^2 - \lambda_3^2 = 0$. Computing a Gröbner basis $\mathcal{G}_{31}$ of the ideal generated by $\mathcal{G}_3 \cup \{\lambda_2^2 - \lambda_3^2\}$, one has that $R^6\lambda_1^2\lambda_3^3$ belongs to $\mathcal{G}_{31}$, which is a contradiction.

Hence the Lie algebra structure (3.2) does not support any $R[\rho]$ -Einstein metric. $\square$

4.2. Left-invariant $R[\rho]$ -Einstein metrics in $\mathbb{R}e_4 \ltimes E(1, 1)$ and $\mathbb{R}e_4 \ltimes E(2)$.

Lemma 8. *Any left-invariant $R[\rho]$ -Einstein metric in $\mathbb{R}e_4 \ltimes E(1, 1)$ or $\mathbb{R}e_4 \ltimes E(2)$ is Einstein.*

Proof. Let $R[\rho]_0$ be the trace-free tensor $R[\rho]_0 = R[\rho] - \frac{1}{4}\|\rho\|^2g$. A straightforward calculation shows that the components of $R[\rho]_0$ for any left-invariant metric (3.4)

are

$$\begin{aligned} 16R^4R[\rho]_{011} &= \mathfrak{R}_{11}, & 8R^4R[\rho]_{012} &= \mathfrak{R}_{12}, & 8R^4R[\rho]_{013} &= \mathfrak{R}_{13}, & 8R^3R[\rho]_{014} &= \mathfrak{R}_{14}, \\ 16R^4R[\rho]_{022} &= \mathfrak{R}_{22}, & 8R^4R[\rho]_{023} &= \mathfrak{R}_{23}, & 8R^3R[\rho]_{024} &= \mathfrak{R}_{24}, & 16R^4R[\rho]_{033} &= \mathfrak{R}_{33}, \\ 4R^3R[\rho]_{034} &= \mathfrak{R}_{34}, & 16R^4R[\rho]_{044} &= \mathfrak{R}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{R}_{ij}$ are given by

$$\begin{aligned} \mathfrak{R}_{11} &= 16b^4 - 7C^4 - 10C^2D^2 - 3D^4 - 14C^2R^2 - 7R^4 + 8C^2R^2\lambda_2 - 4D^2R^2\lambda_2 + 8R^4\lambda_2 \\ &\quad + 4C^2R^2\lambda_2^2 + 2D^2R^2\lambda_2^2 + 10R^4\lambda_2^2 - 16R^4\lambda_2^3 + 5R^4\lambda_2^4 + 8AbCD(-1 + 2\lambda_2) \\ &\quad - 2b^2(9C^2 - 3D^2 + 8R^2(-1 + \lambda_2)\lambda_2) + A^4(-1 + \lambda_2)^2(-7 - 6\lambda_2 + 5\lambda_2^2) \\ &\quad + 2A^2(-7R^2 + 8b^2(-1 + \lambda_2) - 2D^2\lambda_2 + 8R^2\lambda_2 + D^2\lambda_2^2 + 10R^2\lambda_2^2 - 16R^2\lambda_2^3 + 5R^2\lambda_2^4 \\ &\quad + C^2(-7 + 4\lambda_2 + 2\lambda_2^2)), \\ \mathfrak{R}_{12} &= -(A^2CD(-1 + \lambda_2)^2 + 6A^3b(-1 + \lambda_2)^3 + CD(12b^2 + 2C^2 + 2D^2 + R^2 - 2R^2\lambda_2 + R^2\lambda_2^2) \\ &\quad + Ab(-7D^2 - 6R^2 + 8b^2(-1 + \lambda_2) + 5D^2\lambda_2 + 18R^2\lambda_2 - 18R^2\lambda_2^2 + 6R^2\lambda_2^3 + C^2(-5 + 7\lambda_2))), \\ \mathfrak{R}_{13} &= -4A^3D(-1 + \lambda_2)^2\lambda_2 + 4A^2bC(2 - 3\lambda_2 + \lambda_2^2) + bC(24b^2 + 7C^2 + 7D^2 + 7R^2 + 4R^2\lambda_2 \\ &\quad - 11R^2\lambda_2^2) - AD(-2D^2 + 4D^2\lambda_2 + 4R^2\lambda_2 - 8R^2\lambda_2^2 + 4R^2\lambda_2^3 + 4b^2(1 + \lambda_2) + C^2(-1 + 3\lambda_2)), \\ \mathfrak{R}_{14} &= -4A^2D(-1 + \lambda_2)^2\lambda_2 + AbC(1 - 16\lambda_2 + 15\lambda_2^2) + D(C^2 + 2D^2 - 3C^2\lambda_2 - 4D^2\lambda_2 \\ &\quad - 4R^2\lambda_2 + 8R^2\lambda_2^2 - 4R^2\lambda_2^3 + 2b^2(1 + \lambda_2)), \\ \mathfrak{R}_{22} &= 16b^4 - 3C^4 - 10C^2D^2 - 7D^4 + 2C^2R^2 + 4D^2R^2 + 5R^4 + 2b^2(3C^2 - 9D^2 \\ &\quad + 8R^2(-1 + \lambda_2)) + 8AbCD(-2 + \lambda_2) - 4C^2R^2\lambda_2 + 8D^2R^2\lambda_2 - 16R^4\lambda_2 - 14D^2R^2\lambda_2^2 \\ &\quad + 10R^4\lambda_2^2 + 8R^4\lambda_2^3 - 7R^4\lambda_2^4 - A^4(-1 + \lambda_2)^2(-5 + 6\lambda_2 + 7\lambda_2^2) - 2A^2(C^2(-1 + 2\lambda_2) \\ &\quad + D^2(-2 - 4\lambda_2 + 7\lambda_2^2) + (-1 + \lambda_2)(8b^2\lambda_2 + R^2(5 - 11\lambda_2 - \lambda_2^2 + 7\lambda_2^3))), \\ \mathfrak{R}_{23} &= 4A^3C(-1 + \lambda_2)^2 + 4A^2bD(1 - 3\lambda_2 + 2\lambda_2^2) + bD(24b^2 + 7C^2 + 7D^2 - 11R^2 + 4R^2\lambda_2 \\ &\quad + 7R^2\lambda_2^2) + AC(3D^2 + 4R^2 - 2C^2(-2 + \lambda_2) - D^2\lambda_2 - 8R^2\lambda_2 + 4R^2\lambda_2^2 + 4b^2(1 + \lambda_2)), \\ \mathfrak{R}_{24} &= 4A^2C(-1 + \lambda_2)^2 + AbD(15 - 16\lambda_2 + \lambda_2^2) - C(-3D^2 - 4R^2 + 2C^2(-2 + \lambda_2) + D^2\lambda_2 \\ &\quad + 8R^2\lambda_2 - 4R^2\lambda_2^2 + 2b^2(1 + \lambda_2)), \\ \mathfrak{R}_{33} &= -48b^4 + 5C^4 + 10C^2D^2 + 5D^4 + 10C^2R^2 - 8D^2R^2 + 5R^4 - 2b^2(C^2 + D^2 \\ &\quad - 8R^2(-1 + \lambda_2)^2) + 12AbCD(-1 + \lambda_2) + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 + 4R^4\lambda_2 - 8C^2R^2\lambda_2^2 \\ &\quad + 10D^2R^2\lambda_2^2 - 18R^4\lambda_2^2 + 4R^4\lambda_2^3 + 5R^4\lambda_2^4 - A^4(-1 + \lambda_2)^2(3 + 2\lambda_2 + 3\lambda_2^2) + 2A^2(2D^2 + R^2 \\ &\quad - 8b^2(-1 + \lambda_2)^2 - 4D^2\lambda_2 + 4R^2\lambda_2 + D^2\lambda_2^2 - 10R^2\lambda_2^2 + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(1 - 4\lambda_2 + 2\lambda_2^2)), \\ \mathfrak{R}_{34} &= -(-6bCD(-1 + \lambda_2) + 2A^3(-1 + \lambda_2)^2 + A(-3D^2 + 2R^2 + 8b^2(-1 + \lambda_2)^2 + 3D^2\lambda_2 \\ &\quad + 2D^2\lambda_2^2 - 4R^2\lambda_2^2 + 2R^2\lambda_2^4 + C^2(2 + 3\lambda_2 - 3\lambda_2^2))), \\ \mathfrak{R}_{44} &= 16b^4 + 5C^4 + 10C^2D^2 + 5D^4 + 14b^2(C^2 + D^2) + 2C^2R^2 + 4D^2R^2 - 3R^4 \\ &\quad - 36AbCD(-1 + \lambda_2) - 8C^2R^2\lambda_2 - 8D^2R^2\lambda_2 + 4R^4\lambda_2 + 4C^2R^2\lambda_2^2 + 2D^2R^2\lambda_2^2 - 2R^4\lambda_2^2 \\ &\quad + 4R^4\lambda_2^3 - 3R^4\lambda_2^4 + A^4(-1 + \lambda_2)^2(5 + 14\lambda_2 + 5\lambda_2^2) + 2A^2(-4D^2 + R^2 + 16b^2(-1 + \lambda_2)^2 \\ &\quad + 2D^2\lambda_2 + 4R^2\lambda_2 + 5D^2\lambda_2^2 - 10R^2\lambda_2^2 + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(5 + 2\lambda_2 - 4\lambda_2^2)). \end{aligned}$$

Let $\mathcal{I} \subset \mathbb{R}[A, b, \lambda_2, C, D, R]$ be the ideal generated by the polynomials $\{\mathfrak{R}_{ij}\}$ and compute a Gröbner basis $\mathcal{G}_0 \subset \mathcal{I}$ with respect to the lexicographic order. Since the polynomial $\mathbf{g}_0 = D^3(C^2 + D^2)^2 \in \mathcal{G}_0$ one has that $D = 0$. Computing a Gröbner basis $\mathcal{G}_{01}$ of the ideal $\mathcal{G}_0 \cup \{D\}$, one has that $C^5\lambda_2^2 \in \mathcal{G}_{01}$, and thus $C = 0$. Once again, compute a Gröbner basis $\mathcal{G}_{011}$ of the ideal $\mathcal{G}_{01} \cup \{C\}$ to see that $Ab^4 \in \mathcal{G}_{011}$. Therefore either $A = 0$ or $b = 0$. We analyze both possibilities separately.

4.2.1. $A = 0$. Compute a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{G}_{011} \cup \{A\}$ to see that $R^4(\lambda_2 - 1)^4(\lambda_2 + 1) \in \mathcal{G}_1$. Hence $\lambda_2 = \pm 1$. Setting $\lambda_2 = 1$. If $\lambda_2 = 1$, then the $R[\rho]$ -Einstein equations reduce to $b = 0$, in which case the metric is Einstein. Moreover, if $\lambda_2 = -1$, then the $R[\rho]$ -Einstein equations reduce to $b^2 = R^2$ and the metric is Einstein.

4.2.2. $b = 0$. Compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{G}_{011} \cup \{b\}$ to see that $R^2(A^2 + R^2)(\lambda_2^2 - 1)^2 \in \mathcal{G}_2$. Hence $\lambda_2 = \pm 1$. If $\lambda_2 = 1$, then the resulting metric is Einstein while in the case $\lambda_2 = -1$ the $R[\rho]$ -Einstein equations become $\frac{1}{R^4}(A^2 + R^2)^2 = 0$ which has no solutions with $R > 0$. $\square$

4.3. Left-invariant $R[\rho]$ -Einstein metrics in $\mathbb{R}e_4 \ltimes H^3$.

Lemma 9. *Any left-invariant $R[\rho]$ -Einstein metric in $\mathbb{R}e_4 \ltimes H^3$ is Einstein.*

Proof. Let $R[\rho]_0$ be the trace-free tensor $R[\rho]_0 = R[\rho] - \frac{1}{4}\|\rho\|^2 g$. A straightforward calculation shows that the components of $R[\rho]_0$ for any left-invariant metric (3.6) are

$$\begin{aligned} 16R^4R[\rho]_{011} &= \mathfrak{R}_{11}, & 4R^4R[\rho]_{012} &= \mathfrak{R}_{12}, & 8R^4R[\rho]_{013} &= \mathfrak{R}_{13}, & 4R^3R[\rho]_{014} &= \mathfrak{R}_{14}, \\ 16R^4R[\rho]_{022} &= \mathfrak{R}_{22}, & 8R^4R[\rho]_{023} &= \mathfrak{R}_{23}, & 4R^3R[\rho]_{024} &= \mathfrak{R}_{24}, & 16R^4R[\rho]_{033} &= \mathfrak{R}_{33}, \\ 8R^3R[\rho]_{034} &= \mathfrak{R}_{34}, & 16R^4R[\rho]_{044} &= \mathfrak{R}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{R}_{ij}$ are given by

$$\begin{aligned} \mathfrak{R}_{11} &= 16a^4 - 48d^4 - 16d^2F^2 - 3F^4 + 12cdFH - 2d^2H^2 + 2F^2H^2 + 5H^4 \\ &\quad - 2c^2(4d^2 + F^2 + H^2) + 16d^2R^2 + 2F^2R^2 + 10H^2R^2 + 5R^4 - 2a^2(4c^2 + 8d^2 - 3F^2 + 8R^2) \\ &\quad + 4a(4c^2d - cFH - 2d(8d^2 + F^2 + 3H^2 - R^2)), \\ \mathfrak{R}_{12} &= 8a^3c + a^2(-4cd + 2FH) - cd(8d^2 + 2F^2 + 3H^2 - 3R^2) + 2FH(d^2 + F^2 + H^2 + R^2) \\ &\quad + a(-6dFH + c(4d^2 + 3F^2 + 2H^2 - 3R^2)), \\ \mathfrak{R}_{13} &= 8a^3H + a^2(-8cF + 12dH) - cF(8d^2 + 2F^2 + 2H^2 + R^2) + aH(28d^2 + 5F^2 \\ &\quad + 6(H^2 + R^2)) + dH(24d^2 + 8F^2 + 7(H^2 + R^2)), \\ \mathfrak{R}_{14} &= -a^2F + 8d^2F - 3cdH + 2a(5dF + 3cH) + 2F(F^2 + H^2 + R^2), \\ \mathfrak{R}_{22} &= -48a^4 - 64a^3d + 16d^4 + 5F^4 + 4cdFH + 6d^2H^2 + 2F^2H^2 - 3H^4 - 2c^2(4d^2 + F^2 \\ &\quad + H^2) - 16d^2R^2 + 10F^2R^2 + 2H^2R^2 + 5R^4 - 2a^2(4c^2 + 8d^2 + F^2 + 8H^2 - 8R^2) \\ &\quad + 4a(4c^2d - 3cFH - 2d(3F^2 + H^2 - R^2)), \\ \mathfrak{R}_{23} &= 24a^3F + 8d^3F + 8cd^2H + 4a^2(7dF + 2cH) + cH(2F^2 + 2H^2 + R^2) + dF(6F^2 + 5H^2 \\ &\quad + 6R^2) + aF(12d^2 + 7F^2 + 8H^2 + 7R^2), \\ \mathfrak{R}_{24} &= 6cdF - 8a^2H - a(3cF + 10dH) + H(d^2 - 2(F^2 + H^2 + R^2)), \\ \mathfrak{R}_{33} &= -(-16a^4 - 32a^3d - 16d^4 + 16d^2F^2 + 7F^4 - 20cdFH + 18d^2H^2 + 14F^2H^2 + 7H^4 \\ &\quad + 2c^2(4d^2 + F^2 + H^2) + 2a^2(4c^2 - 8d^2 + 9F^2 + 8H^2) - 4a(4c^2d + 8d^3 - 5cFH \\ &\quad - 4d(F^2 + H^2)) + 14F^2R^2 + 14H^2R^2 + 7R^4), \\ \mathfrak{R}_{34} &= 4a^2c - dFH + a(-8cd + FH) + c(4d^2 + F^2 + H^2), \\ \mathfrak{R}_{44} &= 16a^4 + 32a^3d + 16d^4 + 32d^2F^2 + 5F^4 - 36cdFH + 14d^2H^2 + 10F^2H^2 + 5H^4 \\ &\quad + 6c^2(4d^2 + F^2 + H^2) + 2a^2(12c^2 + 8d^2 + 7F^2 + 16H^2) + 2F^2R^2 + 2H^2R^2 - 3R^4 \\ &\quad + 4a(-12c^2d + 9cFH + 4d(2d^2 + 3F^2 + 3H^2 - R^2)). \end{aligned}$$

Therefore $\mathbb{R}e_4 \ltimes H$ is $R[\rho]$ -Einstein if and only if the system of equations $\{\mathfrak{R}_{ij} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, H, F, R]$ be the ideal generated by the polynomials

$\{\mathfrak{R}_{ij}\}$ and compute a Gröbner basis $\mathcal{G}_0 \subset \mathcal{I}$ with respect to the lexicographic order. Since the polynomial $\mathbf{g}_0 = FHR^2(F^2 + H^2 + R^2)(2H^2 + R^2)^3 \in \mathcal{G}_0$ one has that $FH = 0$. Computing a Gröbner basis $\mathcal{G}_{01}$ of the ideal $\mathcal{G}_0 \cup \{FH\}$ with respect to the lexicographic order, one has that the polynomials $HR^2(H^2 + R^2)^2$ and $FR^2(F^2 + R^2)^2$ belong to $\mathcal{G}_{01}$, and hence $F = H = 0$. Computing again a Gröbner basis $\mathcal{G}_{011}$ of the ideal $\mathcal{G}_{01} \cup \{F, H\}$ with respect to the lexicographic order, one has that the polynomial $(4d^2 - R^2)R^6 \in \mathcal{G}_{011}$. Once again we compute a Gröbner basis $\mathcal{G}_{0111}$ of the ideal $\mathcal{G}_{011} \cup \{(4d^2 - R^2)\}$ and get that the polynomial $(a - d)R^4 \in \mathcal{G}_{0111}$. Thus $a = d$ and the resulting metric is Einstein. $\square$

4.4. Left-invariant $R[\rho]$ -Einstein metrics in $\mathbb{R}e_4 \ltimes \mathbb{R}^3$.

Lemma 10. *Any left-invariant $R[\rho]$ -Einstein metric on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ is Einstein.*

Proof. We proceed as in Lemma 6 and consider a left-invariant metric on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ given as in (3.8). Then the components of the trace-free tensor $R[\rho]_0$ become

$$\begin{aligned} 2R^4R[\rho]_{011} &= \mathfrak{R}_{11}, & R^4R[\rho]_{012} &= \mathfrak{R}_{12}, & R^4R[\rho]_{013} &= \mathfrak{R}_{13}, & R[\rho]_{014} &= 0, \\ 2R^4R[\rho]_{022} &= \mathfrak{R}_{22}, & R^4R[\rho]_{023} &= \mathfrak{R}_{23}, & R[\rho]_{024} &= 0, & 2R^4R[\rho]_{033} &= \mathfrak{R}_{33}, \\ R[\rho]_{034} &= 0, & 2R^4R[\rho]_{044} &= \mathfrak{R}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{R}_{ij}$ are given by

$$\begin{aligned} \mathfrak{R}_{11} &= -(-a^4 + b^2f^2 + f^4 + f^2h^2 + f^3p - 2fh^2p + c^2p^2 + 2f^2p^2 + h^2p^2 + fp^3 + p^4 \\ &\quad + a^3(f + p) + a^2(b^2 + c^2 - 2f^2 + fp - 2p^2) - a(2b^2f + f^3 + 2c^2p + f^2p + fp^2 + p^3)), \\ \mathfrak{R}_{12} &= -(b(a - f)(a^2 - af + f^2 + p^2)), \\ \mathfrak{R}_{13} &= -(c(a - p)(a^2 + f^2 - ap + p^2)), \\ \mathfrak{R}_{22} &= -(a^4 + b^2f^2 - f^4 + f^2h^2 + f^3p - 2fh^2p + c^2p^2 - 2f^2p^2 + h^2p^2 - fp^3 + p^4 \\ &\quad + a^3(-f + p) + a^2(b^2 + c^2 - 2f^2 - fp + 2p^2) + a(-2b^2f + f^3 - 2c^2p + f^2p - fp^2 + p^3)), \\ \mathfrak{R}_{23} &= -(h(f - p)(a^2 + f^2 - fp + p^2)), \\ \mathfrak{R}_{33} &= -(a^4 + b^2f^2 + f^4 + f^2h^2 + a^3(f - p) - f^3p - 2fh^2p + c^2p^2 - 2f^2p^2 + h^2p^2 + fp^3 \\ &\quad - p^4 + a^2(b^2 + c^2 + 2f^2 - fp - 2p^2) + a(-2b^2f + f^3 - 2c^2p - f^2p + fp^2 + p^3)), \\ \mathfrak{R}_{44} &= a^4 + 3b^2f^2 + f^4 + 3f^2h^2 + f^3p - 6fh^2p + 3c^2p^2 - 2f^2p^2 + 3h^2p^2 + fp^3 + p^4 \\ &\quad + a^3(f + p) + a^2(3b^2 + 3c^2 - 2f^2 - fp - 2p^2) + a(-6b^2f + f^3 - 6c^2p - f^2p - fp^2 + p^3). \end{aligned}$$

Hence a left-invariant metric (3.8) is $R[\rho]$ -Einstein if and only if the system of polynomial equations $\{\mathfrak{R}_{ij} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[a, b, c, f, h, p, R]$ be the ideal generated by the polynomials $\{\mathfrak{R}_{ij}\}$ and compute a Gröbner basis $\mathcal{G}_0$ with respect to the lexicographic order. Since the polynomial $\mathbf{g}_0 = p^7(f^3 - p^3)$ belongs to $\mathcal{G}_0$, one has that either $p = 0$ or $p = f$. We analyze the two cases separately.

4.4.1. $p = 0$. Compute a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{G}_0 \cup \{p\}$ to see that $f^7 \in \mathcal{G}_1$, and thus $f = 0$. Compute again a Gröbner basis $\mathcal{G}_{11}$ of the ideal $\mathcal{G}_1 \cup f$. It now becomes that the new Gröbner basis is given by

$$\mathcal{G}_{11} = \{p, f, a^2(b^2 + c^2), a^3c, a^2b, a^4\}.$$

Hence $a = 0$ and the resulting metric is flat.

4.4.2. $p = f$, $p \neq 0$. Compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{G}_0 \cup \{p - f\}$ to see that $p^6(a - p) \in \mathcal{G}_2$. Since $p \neq 0$, one has $p = a$. The resulting metric is now flat, which finishes the proof. $\square$

5. PROOF OF THEOREM 1

Assertion (3) in Theorem 1 follows at once from the results in Section 4.

In order to proof Assertion (1), we recall that the $(1, 3)$ -Weyl curvature operator is invariant by conformal transformations. Indeed, if two Riemannian metrics are conformally equivalent (i.e., $\tilde{g} = e^{2\sigma}g$), then their Weyl curvature operators $\widetilde{\mathcal{W}}$ and $\mathcal{W}$ are equal. The converse does not hold in general, but it is true in dimension four (see [6]). Note that the corresponding $(0, 4)$ -Weyl curvature tensors and their norms satisfy $\widetilde{W} = e^{2\sigma}W$ and $\|\widetilde{W}\|_{\tilde{g}}^2 = e^{-4\sigma}\|W\|_g^2$. Hence, if the conformally-related metrics are both homogeneous (since $\|W\|_g^2$ and $\|\widetilde{W}\|_{\tilde{g}}^2$ are both constant in this case), one has that either g and $\tilde{g}$ are homothetic or otherwise both metrics are locally conformally flat. Furthermore, since homogeneous locally conformally flat metrics are symmetric [14], non-symmetric conformally related homogeneous metrics are necessarily homothetic.

It was shown in [1] that any non-symmetric homogeneous weakly-Einstein metric is determined by the Lie algebra structure

$$(5.9) \quad [e_4, e_1] = \alpha e_1, \quad [e_4, e_2] = -\alpha e_2 - \beta e_3, \quad [e_4, e_3] = \beta e_2 - \alpha e_3,$$

where $\alpha \neq 0$ and β are constants and $\{e_1, \dots, e_4\}$ is an orthonormal basis. Now, a explicit calculation shows that the $(1, 3)$ -Weyl curvature operator of any metric (5.9) satisfies

$$\begin{aligned} \mathcal{W}(e_1, e_2) &= \frac{1}{3}\alpha^2(E^1_2 - E^2_1), & \mathcal{W}(e_1, e_3) &= \frac{1}{3}\alpha^2(E^1_3 - E^3_1), \\ \mathcal{W}(e_1, e_4) &= \frac{2}{3}\alpha^2(E^4_1 - E^1_4), & \mathcal{W}(e_2, e_3) &= \frac{2}{3}\alpha^2(E^3_2 - E^2_3), \\ \mathcal{W}(e_2, e_4) &= \frac{1}{3}\alpha^2(E^2_4 - E^4_2), & \mathcal{W}(e_3, e_4) &= \frac{1}{3}\alpha^2(E^3_4 - E^4_3) \end{aligned}$$

where E^i_j denotes the matrix with 1 in the position (i, j) and zero otherwise.

Hence the Lie group determined by (5.9) is homothetic (but not isomorphically homothetic) to the Lie group determined by the Lie algebra

$$(5.10) \quad [e_4, e_1] = \alpha e_1, \quad [e_4, e_2] = -\alpha e_2, \quad [e_4, e_3] = -\alpha e_3,$$

Finally, the Lie groups determined by (5.10) are all isomorphically homothetic to the Lie group structure in Theorem 1-(1), thus showing Assertion (1).

The structure corresponding to Theorem 1-(2.a) has zero scalar curvature, while the scalar curvature in the cases (2.b) and (2.c) is strictly negative, so the Lie group corresponding to Assertion (2.a) cannot be homothetic to any of (2.b) or (2.c) in Theorem 1. Furthermore, the structure defined by Assertion (2.b) satisfies

$$\tau = -1, \quad \|\rho\|^2 = 1, \quad \|R\|^2 = 3.$$

On the other hand, replacing the metric $\langle \cdot, \cdot \rangle_\alpha$ in Assertion (2.c) by the homothetic one $\langle \cdot, \cdot \rangle_\alpha^* = \frac{2(1+\alpha+\alpha^2)^2}{(1+\alpha)^2} \langle \cdot, \cdot \rangle_\alpha$, a straightforward calculation shows that

$$\tau_\alpha = -1, \quad \|\rho_\alpha\|^2 = 1, \quad \|R_\alpha\|^2 = \frac{3 + \alpha(1+\alpha)(9 + \alpha(1+\alpha)(7 + 3\alpha(1+\alpha)))}{(1 + \alpha + \alpha^2)^3}.$$

Now, any homothety between the metric in Assertion (2.b) and any of the metrics in Assertion (2.c) must induce an isometry between the metric (2.b) and some metric $\langle \cdot, \cdot \rangle_\alpha^*$. Hence it must preserve the norm of the curvature tensor, and thus $\|R_\alpha\|^2 = 3$ for some α . A straightforward calculation now shows that $\|R_\alpha\|^2 = 3$ if and only if either $\alpha = 0$ or $\alpha = -1$ and none of these cases may occur. Therefore

the structure in Assertion (2.b) cannot be homothetic to any of the structures given by Assertion (3.c). $\square$

Remark 11. The solvable Lie groups in Theorem 1 are diffeomorphic to $\mathbb{R}^4$. More precisely

- (1) The structure constants in Assertion (1) are equivalent to

$$de^4 = 0, \quad de^1 = e^1 \wedge e^4, \quad de^2 = -e^2 \wedge e^4, \quad de^3 = -e^3 \wedge e^4,$$

where $\{e^1, \dots, e^4\}$ is the dual basis of $\{e_1, \dots, e_4\}$. Hence $\mathbb{R} \ltimes \mathbb{R}^3$ in Assertion (1) is isometric to $(\mathbb{R}^4, g)$ with coordinates (x, y, z, t) and metric

$$g = dt^2 + e^{-2t}dx^2 + e^{2t}(dy^2 + dz^2).$$

- (2) The structure constants in Assertion (2.b) are equivalent to

$$de^4 = 0, \quad de^1 = \frac{1}{2}e^4 \wedge e^1, \quad de^2 = -\frac{1}{2}e^4 \wedge e^2, \quad de^3 = e^2 \wedge e^1.$$

Setting $e^4 = dt$, $e^1 = e^{-\frac{1}{2}t}dx$, $e^2 = e^{\frac{1}{2}t}dy$ and $e^3 = \frac{1}{2}ydx - \frac{1}{2}xdy + dz$, the Lie group in Assertion (2.b) is isometric to $\mathbb{R}^4$ with metric tensor

$$g = dt^2 + e^{-t}dx^2 + e^t dy^2 + \left(\frac{1}{2}ydx - \frac{1}{2}xdy + dz\right)^2.$$

- (3) The structure constants in Assertion (2.c) are equivalent to We should try to complete this part

Remark 12. Investigate the existence of relevant properties for the metrics in Theorem 1: are they algebraic Ricci solitons?, etc.

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